COMPLETE INTERSECTIONS IN RATIONAL HOMOTOPY THEORY.

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Abstract. We investigate various homotopy invariant formulations of commutative algebra in the context of rational homotopy theory. The main subject is the complete intersection condition, where we show that a growth condition implies a structure theorem and that modules have multiply periodic resolutions.

Contents

1. Introduction

1.A. **Background.** It has been very fruitful to adapt the definitions of commutative algebra so that they apply in homotopy theory. The original motivation is that it is useful to study a space X through a ring of functions, and for our purposes we will think of the ring $C^*(X;k)$ of cochains on X. Of course, if the analogy is to be accurate, we need a commutative model for $C^*(X;k)$, and if it is to be effective we need to render the definitions homotopy invariant.

The prime example of this is the connection between rational homotopy theory and rational differential graded algebras (DGAs), but the availability of good models for ring spectra has led to other useful examples in positive characteristic. The emphasis in classical rational homotopy theory has been on finite complexes and calculation, whereas one of the themes in characteristic p has been to consider classifying spaces of compact Lie groups where the

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natural finiteness condition is that the cohomology rings in question are Noetherian. The purpose of the present paper is to take the ideas developed for compact Lie groups and investigate them in the much more accessible context of rational homotopy theory. From one point of view this is the process of generalizing classical results [18] from the case when $H^*(X;\mathbb{Q})$ is finite dimensional to the case when it is Noetherian, and given the available tools of rational homotopy theory, this is reasonably straightforward. From another point of view this is an opportunity to give new and accessible examples of the theory, and to test expectations in a context where complete calculation is often possible. Finally, the work suggests a number of questions we may translate through the mirror [6] to local algebra, and we plan to investigate these in future.

1.B. **Contents.** On the commutative algebra side we restrict attention to commutative, local, Noetherian rings. On the topological side, we restrict attention to simply connected, rational spaces X with $H^*(X)$ Noetherian.

We begin by considering analogues of regular and Gorenstein local rings. Both are already well-known in rational homotopy theory, but it gives us an opportunity to introduce some terminology and to express things in a convenient language. For example, we emphasize the importance of homotopy invariant finiteness conditions and Morita theory. The regular spaces X are precisely those which are finite products of even Eilenberg-MacLane spaces. There are enormous numbers of Gorenstein spaces, and they include manifolds and all finite Postnikov systems.

In the classical literature of rational homotopy theory, Gorenstein *duality* does not seem to be a familiar phenomenon except in the zero-dimensional cases (Poincaré Duality). We take the opportunity in Appendix A to explain how the Local Cohomology Theorem from [13] gives a Gorenstein duality statement in general. From the point of view of rational homotopy theory it shows (for example) that if X is any finite Postnikov system and $H^*(X)$ is Cohen-Macaulay, it is automatically Gorenstein. Furthermore, without any hypothesis on the depth, $H^*(X)$ is generically Gorenstein. From the point of view of homotopy invariant commutative algebra, it gives an extremely rich and flexible source of examples.

The main subject of the paper is a study of the complete intersection (ci) condition. We give a number of homotopy invariant definitions of ci spaces, corresponding to different aspects of the ci condition. These have very different characters, so it is striking that we are able to show that in the rational context they are all equivalent. The *structural* condition in commutative algebra is that a ci ring is a quotient of a regular local ring by a regular sequence. We say that a simply connected rational space X is sci if it is formed from a finite product of even Eilenberg-MacLane spaces by iterated spherical fibrations (all definitions are given precisely in Section 7). Secondly, Benson and the first author [9] introduced a finiteness condition (zci) on the category of modules analogous to requiring all modules to have eventually multiply periodic resolutions. This is a strengthening of the condition in [14]. In this paper we needed to relax the zci condition to two new finiteness conditions, the eci and the nci conditions. Finally, in commutative algebra there is the *growth* condition that $\mathrm{Ext}_R^*(k,k)$ has polynomial growth (equivalent to the structural condition by Gulliksen's theorem); the condition on a rational simply connected space is the growth condition (gci) that $H_*(\Omega X)$ has polynomial growth.

Most remarkable of the equivalences, perhaps, is the fact that the growth condition implies a structure theorem: X is gci if and only if there is a fibration

$$F \longrightarrow X \longrightarrow KV,$$

where KV is a finite product of even Eilenberg-MacLane spaces and $\pi_*(F)$ is entirely in odd degrees. Amongst these spaces, those in which F has trivial k-invariants, so that F is a product of odd spheres, are the ones with pure Sullivan models.

Another unexpected phenomenon is the importance of the Noetherian condition. On the one hand, an iterated spherical fibration over a product of even Eilenberg-MacLane spaces is obviously Noetherian. One might naively think that requiring $H_*(\Omega X)$ to have polynomial growth would be enough without requiring $H^*(X)$ to be Noetherian, but in fact the Milnor-Moore theorem shows that this just means $\pi_*(X)$ is finite dimensional. It is very striking that the Noetherian condition is sufficient to give a structure theorem, and we are grateful to N.P.Strickland for a timely remark. We also thank S.B.Iyengar for comments.

1.C. **The layout of the paper.** After summarizing conventions in Section 2, we begin in Section 3 by giving a brief summary of the results and terminology we need from rational homotopy theory. Next, in Section 4 we describe the Morita theory for moving between $C^*(X)$ and $C_*(\Omega X)$, and some results on cellularization from [13]. We are then in a position to consider rational DGAs in parallel with rational spaces. In a series of sections we describe the definitions for rational DGAs and in particular for Sullivan models of rational spaces. In Section 5 we consider regular rings and spaces, and in Appendix A we discuss Gorenstein spaces and Gorenstein duality.

From Section 6 onwards, our main concern is for complete intersections. First, Section 6 discusses the centre of a derived category, and how bimodules and Hochschild cohomology give elements of the centre. Section 7 introduces the definitions designed to capture various aspects of hypersurface and ci spaces, which later sections show to be equivalent. Section 8 takes the structural definition, and shows that any sci space has a standard form. Section 9 gives the elementary argument that zci spaces satisfy the gci growth condition. Section 10 shows that sci spaces all have eventually multiply periodic module theories. In Section 11 we calculate the Hochschild cohomology of all pure sci spaces relative to their regular base and use the result to show they are zci. Finally, and perhaps most interestingly, in Section 12 we show that the growth condition alone is enough to show that a space has the standard sci form. Section 13 gives a number of explicit examples illustrating the phenomena we have studied, and showing that the various classes of spaces are distinct. The final section explores the role of the Noetherian condition further, and gives a characterisation of the polynomial growth of $H_*(\Omega X)$ when we do not require $H^*(X)$ to be Noetherian in the same style as the multiply periodic resolution property for ci spaces.

2. Conventions.

2.A. **Terminology for triangulated categories.** Recall that an object X of a triangulated category $\mathcal{T}$ is called *small* if the natural map

$$\bigoplus_i [X, Y_i] \longrightarrow [X, \bigvee_i Y_i]$$

is an isomorphism for any set of objects Y_i.

A *thick* subcategory of $\mathcal{T}$ is a full subcategory closed under completion of triangles and taking retracts. We write $\mathrm{thick}(X)$ for the smallest thick subcategory containing X, and if $A \in \mathrm{thick}(X)$ we also say 'X *finitely builds* A' and write $X \models A$.

A *localizing* subcategory of $\mathcal{T}$ is a thick subcategory which is also closed under taking arbitrary coproducts. We write $\mathrm{loc}(X)$ for the smallest localizing subcategory containing X, and if $A \in \mathrm{loc}(X)$ we also say 'X *builds* A' and write $X \vdash A$.

Following [14] we say that X is *virtually small* if $\mathrm{thick}(X)$ contains a non-trivial small object W, and we say that any such W is a *witness* for the fact that X is virtually small.

2.B. **Grading conventions.** We will have cause to discuss homological and cohomological gradings. Our experience is that this a frequent source of confusion, so we adopt the following conventions. First, we refer to lower gradings as *degrees* and upper gradings as *codegrees*. As usual, one may convert gradings to cogradings via the rule $M_n = M^{-n}$. Thus both chain complexes and cochain complexes have differentials of degree -1 (which is to say, of codegree 1). This much is standard. However, since we need to deal with both chain complexes and cochain complexes it is essential to have separate notation for homological suspensions (Σ^i) and cohomological suspensions (Σ_i): these are defined by

$$(\Sigma^i M)_n = M_{n-i} \text{ and } (\Sigma_i M)^n = M^{n-i}.$$

Thus, for example, with reduced chains and cochains of a based space X, we have

$$\tilde{C}_*(\Sigma^i X) = \Sigma^i \tilde{C}_*(X) \text{ and } \tilde{C}^*(\Sigma^i X) = \Sigma_i \tilde{C}^*(X).$$

2.C. **Other conventions.** Unless explicitly stated to the contrary, all coefficients will be in the rational numbers $\mathbb{Q}$, and for a rational vector space V, we write $V^\vee = \mathrm{Hom}_{\mathbb{Q}}(V, \mathbb{Q})$ for the dual vector space.

For brevity we write CGA for *commutative graded algebra* (i.e., an algebra which is commutative in the graded sense that $xy = (-1)^{|x|\cdot|y|} yx$), DGA for *differential graded algebra*, and CDGA for *commutative differential graded algebra*. When we refer to modules over a DGA, we intend differential graded modules unless otherwise stated.

Finally, for a space X, we write $C^*(X)$ for a CDGA model for the cochains on X.

3. Rational homotopy theory.

Rational homotopy theory provides the ideal context to test ideas about homotopy invariant commutative algebra. On the one hand many aspects of commutative algebra are especially simple for $\mathbb{Q}$-algebras and on the other we can appeal to the intuition and structures of homotopy theory.

3.A. **Terminology for commutative differential graded algebras.** We will restrict attention to simply connected $\mathbb{Q}$-algebras of finite type.

If V is a graded rational vector space, we write $\Lambda(V)$ for the free CGA on V. This is a symmetric algebra on V^{ev} tensored with an exterior algebra on V^{od}. A *Sullivan algebra* is a CDGA which is free as a CGA on a simply connected graded vector space V of finite dimension in each degree, and whose differential has the property that if $x \in V^s$ then $dx \in \Lambda(V^{<s})$. It is minimal if in addition d takes values in $\Lambda^+(V)^2$. A Sullivan algebra $(\Lambda(V^{od} \oplus V^{ev}), d)$ is *pure* if $d(V^{od}) \subset \Lambda V^{ev}$ and $d(V^{ev}) = 0$.

A *relative Sullivan algebra* is a map $M \longrightarrow M \rtimes \Lambda(V)$ of CDGAs. Here the underlying CGA of $M \rtimes \Lambda(V)$ is $M \otimes \Lambda(V)$, and the notation expresses the fact that M is a sub-DGA and $\Lambda(V)$ is a quotient.

3.B. **Rational models for simply connected spaces.** Any simply connected rational CW-complex with cohomology finite in each degree is modelled by a simply connected rational CDGA (such as the CDGA of PL polynomial differential forms $\mathcal{A}_{PL}(X)$). Furthermore, any such CDGA has a Sullivan minimal model, unique up to isomorphism. We write $C^*(X)$ for an unspecified CDGA model for the cochains on X. The process of building up a Sullivan algebra degree by degree corresponds to building up a space using a Postnikov tower.

If V is an evenly graded vector space, we write KV for the associated Eilenberg-MacLane space. In principle we could use the same notation when V has an odd summand, but we will not do so. Since odd spheres are rational Eilenberg-MacLane spaces, if W is a graded vector space in odd degrees, we write $S(W)$ for the corresponding Eilenberg-MacLane space.

We say a space X is *pure* if X has pure Sullivan algebra model.

A fibration $E \longrightarrow B$ with fibre F can be modelled by a relative Sullivan algebra $M \rtimes \Lambda(V) \longleftarrow M$ where M models B, $M \rtimes \Lambda(V)$ models E and the fibre F is then modelled by $\Lambda(V)$.

3.C. **Homotopy Lie algebras and the Milnor-Moore theorem.** Recall that $\pi_*(\Omega X)$ is a graded Lie algebra under the Samelson product. More precisely there is a natural bilinear product

$$[\cdot,\cdot] : \pi_i(\Omega X) \times \pi_j(\Omega X) \longrightarrow \pi_{i+j}(\Omega X)$$

which is antisymmetric in the sense that

$$[x,y] = -(-1)^{|x|\cdot|y|}[y,x]$$

and satisfies the graded Jacobi identity

$$(-1)^{|x|\cdot|z|}[x,[y,z]] + (-1)^{|y|\cdot|x|}[y,[z,x]] + (-1)^{|z|\cdot|y|}[z,[x,y]] = 0.$$

One way of forming a graded Lie algebra from an associative algebra A is to define $[x,y] = xy - (-1)^{|x|\cdot|y|}yx$ for homogeneous elements $x, y \in A$. Associated to a graded Lie algebra is a universal associative algebra

$$U(L) = TL/I$$

where TL is the tensor algebra on L and I is the ideal generated by the relations $[x,y] = x \otimes y - (-1)^{|x|\cdot|y|}y \otimes x$ for $x, y \in L$. Henceforth we will generally omit the notation for the tensor product.

The most important algebraic fact about the universal enveloping algebra of a Lie algebra is the Poincaré-Birkoff-Witt theorem stating that if we filter $U(L)$ by tensor length then there is an isomorphism

$$Gr(U(L)) = \Lambda L.$$

In particular, the growth rate of $U(L)$ is the same as that of the symmetric algebra on L^{ev}.

The following theorem makes this relevant to topology.

Theorem 3.1. *(Milnor-Moore* [26]*) If X is a simply connected rational space then*

$$H_*(\Omega X) = U(\pi_*(\Omega X)). \quad \square$$

In particular, we see that $H_*(\Omega X)$ has polynomial growth if and only if $\pi_*(\Omega X)$ is finite dimensional, and in that case the growth is of degree one less than

$$\dim_{\mathbb{Q}}(\pi_{ev}(\Omega X)) = \dim_{\mathbb{Q}}(\pi_{od}(X)).$$

3.D. **Elliptic spaces.** Perhaps for historical reasons, classical rational homotopy theory concentrates on finite complexes, which is to say spaces with $H^*(X)$ finite dimensional. These correspond to 0-dimensional local rings.

A simply connected rational space X is called *elliptic* if $H^*(X)$ and $\pi_*(X)$ are both finite dimensional. It is called *hyperbolic* if $\pi_*(X)$ has exponential growth.

A major theorem of rational homotopy theory is the dichotomy theorem stating that a simply connected rational space with $H^*(X)$ finite dimensional is either elliptic or hyperbolic. In a sense we will make precise, elliptic spaces correspond to 0-dimensionsional complete intersections.

3.E. **Noether normalization.** Polynomial rings on even degree generators play a special role in the theory. To start with, they are *intrinsically formal*: if P is a polynomial ring on even degree generators, then if A is any CDGA with $H^*(A) \cong P$, we have a quasi-isomorphism $A \simeq P$. Indeed, P has a useful universal property: for any CDGA A, and any map $\theta : P \longrightarrow H^*(A)$ of CGAs, a choice of representative cycles for the polynomial generators allows us to realize θ by a map $\tilde{\theta} : P \longrightarrow A$ of CDGAs. Not only are they convenient, we shall see they have a structural role: polynomial rings on even degree generators provide the class of CDGAs corresponding to regular local rings. We think of KV with V even and finite dimensional as a generalization of the rational classifying space of a compact connected Lie group.

Polynomial rings can then be used in the study of general Noetherian rings. Indeed, the Noether normalization theorem states that if R is a Noetherian connected CGA, it is finitely generated as a module over a polynomial subalgebra P on even degree generators. We will repeatedly use the following counterpart of this statement.

Proposition 3.2. *If X is a 1-connected rational space with $H^*(X)$ Noetherian, there is a fibration*

$$F \longrightarrow X \longrightarrow KV$$

of rational spaces where V is even and finite dimensional, and $H^*(F)$ is finite dimensional.

Proof: By Noether normalization, $H^*(X)$ is finite dimensional over a polynomial algebra P on even degree generators. Choosing representative cycles, we have a map $P = KV \longrightarrow C^*(X)$ of CDGAs realizing this map in cohomology. This gives a fibration

$$F \longrightarrow X \longrightarrow KV.$$

To see $H^*(F)$ is finite dimensional, we note that $H^*(X)$ is a finitely generated P-module, and therefore has a finite resolution by finitely generated free P-modules. $\square$

We refer to this fibration as a Noether normalization of X, and to F as a *Noether fibre* of X. The long exact sequence in homotopy shows that the growth of $\pi_*(X)$ is the same as that of $\pi_*(F)$.

Lemma 3.3. (Dichotomy) *For a space X with $H^*(X)$ Noetherian, either $\pi_*(X)$ is finite dimensional or it has exponential growth. The homotopy is finite dimensional if and only if a Noether fibre is elliptic.* $\square$

This motivates the following extension of the notion of elliptic spaces to spaces with Noetherian cohomology.

Definition 3.4. A space X is *gci* (or satisfies the growth condition for a complete intersection) if $H^*(X)$ is Noetherian and $\pi_*(X)$ is finite dimensional.

These spaces are the principal subject of the present paper, and we return to them in Section 7.

3.F. **Some analogies.** At the most basic level, cofibre sequences

$$X \longrightarrow Y \longrightarrow Z$$

of pointed spaces induce (additive) exact sequences

$$C^*(X) \longleftarrow C^*(Y) \longleftarrow C^*(Z)$$

of reduced cochains. On the other hand, fibrations

$$F \longrightarrow E \longrightarrow B$$

of spaces induce (multiplicative) exact sequences

$$C^*(F) \overset{EM}{\simeq} C^*(E) \otimes_{C^*(B)} \mathbb{Q} \longleftarrow C^*(E) \longleftarrow C^*(B)$$

provided $C^*(B)$ is 1-connected so that an Eilenberg-Moore theorem (EM) holds, and $C^*(B) \longrightarrow C^*(E)$ is a relative Sullivan model so that the tensor product is derived.

More generally, a homotopy pullback square

$$\begin{array}{ccc} Z \times_X Y & \longrightarrow & Z \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \end{array}$$

induces a homotopy pushout square

$$\begin{array}{ccc} C^*(Z \times_X Y) & \longleftarrow & C^*(Z) \\ \uparrow & & \uparrow \\ C^*(Y) & \longleftarrow & C^*(X) \end{array}$$

in the sense that

$$C^*(Z \times_X Y) \simeq C^*(Z) \otimes_{C^*(X)} C^*(Y)$$

if X is 1-connected, and one of the maps $C^*(X) \longrightarrow C^*(Z)$ or $C^*(X) \longrightarrow C^*(Y)$ is a relative Sullivan algebra so that the tensor product is derived.

We should also record the Rothenberg-Steenrod theorem stating that for a fibration $F \longrightarrow E \longrightarrow B$ we have equivalences

$$C_*(E) \simeq C_*(F) \otimes_{C_*(\Omega B)} k \text{ and } C^*(E) \simeq \mathrm{Hom}_{C_*(\Omega B)}(k, C^*(F)).$$

4. The Morita context

We have a simply connected rational space of finite type X, and we consider the CDGA $C^*(X)$. We often wish to translate to statements about the DGA $C_*(\Omega X)$. Throughout we work in derived categories of DG-modules such as $\mathbf{D}(C^*(X))$ or $\mathbf{D}(C_*(\Omega X))$, so tensor products and Homs are derived. As mentioned above, we usually refer simply to 'modules' since the requirement that our modules respect the differentials is implicit in the category we work in. The material is adapted from [12, 13].

4.A. **The two algebras.** We need to see first that the $C^*(X)$ (a *commutative* DGA) and $C_*(\Omega X)$ (which will usually not be commutative) determine each other.

Proposition 4.1. *If X is 1-connected, there are equivalences*

$$C_*(\Omega X) \simeq \mathrm{Hom}_{C^*(X)}(\mathbb{Q}, \mathbb{Q})$$

and

$$C^*(X) \simeq \mathrm{Hom}_{C_*(\Omega X)}(\mathbb{Q}, \mathbb{Q})$$

of DGAs.

Proof: The first of these is the Eilenberg-Moore theorem [15] and the second is the Rothenberg-Steenrod theorem [27]. $\square$

4.B. **The adjunction.** The proposition shows that we have an adjoint pair of functors

$$\mathrm{Hom}_{C^*(X)}(\mathbb{Q}, \cdot) : C^*(X)\text{-mod} \rightleftarrows \text{mod-}C_*(\Omega X) : (\cdot) \otimes_{C_*(\Omega X)} \mathbb{Q}\ .$$

This induces an equivalence between subcategories of the derived categories, but it will be enough for us to know we can move between the module categories and to understand one composite.

4.C. **Cellularization.** An object in the derived category of $C^*(X)$-modules is said to be $\mathbb{Q}$-*cellular* if it is built from $\mathbb{Q}$ up to equivalence. A map $M \longrightarrow N$ of $C^*(X)$-modules is a $\mathbb{Q}$-equivalence if

$$\mathrm{Hom}_{C^*(X)}(\mathbb{Q}, M) \longrightarrow \mathrm{Hom}_{C^*(X)}(\mathbb{Q}, N)$$

is a homology isomorphism. A map $M \longrightarrow N$ is $\mathbb{Q}$-*cellular approximation* if it is a $\mathbb{Q}$-equivalence and M is $\mathbb{Q}$-cellular. By the usual formal argument, this is unique up to equivalence, and we write $\mathrm{Cell}_{\mathbb{Q}}(N) \longrightarrow N$ for it.

We will give two models for $\mathbb{Q}$-cellularization, and it will be valuable to know they are equivalent.

4.D. **The Morita model.** The first model comes from the Morita context.

Proposition 4.2. [12, 13] *If $H^*(X)$ is Noetherian, the counit*

$$\mathrm{Hom}_{C^*(X)}(\mathbb{Q}, M) \otimes_{C_*(\Omega X)} \mathbb{Q} \longrightarrow M$$

of the adjunction is $\mathbb{Q}$-cellularization. $\square$

We need only observe that $C^*(X)$ is proxy-regular in the sense of [13]. Since $H^*(X)$ is Noetherian, the Koszul complex associated to a system of parameters provides a proof.

4.E. **The stable Koszul model.** If R is a commutative ring and $I = (x_1, x_2, \ldots, x_r)$ is an ideal, then Grothendieck defines the local cohomology of an R-module N by the formula

$$H^*_I(R; N) = H^*((R \longrightarrow [\frac{1}{x_1}]) \otimes_R (R \longrightarrow [\frac{1}{x_2}]) \otimes_R \cdots \otimes_R (R \longrightarrow [\frac{1}{x_n}]) \otimes_R N),$$

and shows it calculates the right derived functors of I-power torsion when R is Noetherian. We write $H^*_I(R) = H^*_I(R; R)$ for brevity.

We now lift this to DGAs in the usual way. If $x \in H^*(A)$, we write $\Gamma_x A = \mathrm{fibre}(A \longrightarrow A[1/x])$, and if $I = (x_1, x_2, \ldots, x_n)$ is an ideal in $H^*(A)$, for an A-module M we write

$$\Gamma_I M = \Gamma_{x_1} A \otimes_A \Gamma_{x_2} A \otimes_A \cdots \otimes_A \Gamma_{x_n} A \otimes_A M.$$

It turns out that up to equivalence this depends only on the ideal I, and indeed, only on the radical of I. If $I = \mathfrak{m}$ is the maximal ideal we abbreviate this $\Gamma M = \Gamma_{\mathfrak{m}} M$.

Note that ΓM has a filtration from its construction, and that we therefore have a spectral sequence for calculating its homology.

Lemma 4.3. *There is a spectral sequence*

$$H^*_I(H^*(A); H^*(M)) \Rightarrow H^*(\Gamma M). \quad \square$$

Finally, the relevance to us is that this gives another construction of cellularization.

Proposition 4.4. [13, 9.3] *The natural map*

$$\Gamma M \longrightarrow M$$

is $\mathbb{Q}$-cellularization. $\square$

Now we specialize to the case $A = C^*(X)$ to obtain the required equivalence from uniqueness of cellularization.

Corollary 4.5. *There is a natural equivalence*

$$\Gamma M \simeq \mathrm{Hom}_{C^*(X)}(\mathbb{Q}, M) \otimes_{C_*(\Omega X)} \mathbb{Q}. \quad \square$$

5. Regular rings and spaces.

We shall show that the regular spaces are precisely the spaces KV where V is even and finite dimensional. This is straightforward once we have established definitions.

For all classical commutative algebra, we refer the reader to [25].

5.A. **Definitions.** In commutative algebra there are three styles for a definition of a regular local ring: ideal theoretic, in terms of the **g**rowth of the Ext algebra and a **h**omotopy invariant version.

Definition 5.1. (i) A local Noetherian ring R is *regular* if the maximal ideal is generated by a regular sequence.

(ii) A local Noetherian ring R is *g-regular* if $\mathrm{Ext}^*_R(k, k)$ is finite dimensional.

(iii) A local Noetherian ring R is *h-regular* if every finitely generated module is small in $\mathbf{D}(R)$.

It is not hard to see that g-regularity is equivalent to h-regularity or that regularity implies g-regularity. Serre proved that g-regularity implies regularity, so the three conditions are equivalent.

It is not altogether clear what should play the role of finitely generated modules in the more general context. We would like it to include all small objects, and the object $\mathbb{Q}$, and we would like to know that if $\mathbb{Q}$ is small then all objects in the class are small. For the purpose of the present paper, we take

$$\mathcal{FG} := \{M \mid H^*(M) \text{ is a finitely generated } H^*(X)\text{-module}\},$$

and we will show that it has the properties we require.

Definition 5.2. (i) A space X is *s-regular* if there are fibrations

$$S^{n_1} \longrightarrow X_1 \longrightarrow X, S^{n_2} \longrightarrow X_2 \longrightarrow X_1, \ldots, S^{n_d} \longrightarrow X_d \longrightarrow X_{d-1}$$

with $X_d \simeq *$.

(ii) A space X is *g-regular* if $H_*(\Omega X)$ is finite dimensional.

(iii) A space X is *h-regular* if every object of $\mathcal{FG}$ is small in $\mathbf{D}(C^*(X))$.

If X is s-regular, we see $\Omega X_{d-1} \simeq S^{n_d}$, and working back up the sequence of fibrations, we see that X is g-regular. Since $\mathbb{Q} \in \mathcal{FG}$ it follows from Proposition 4.1 that an h-regular space is g-regular. We will establish the reverse implication by classifying g-regular spaces.

Remark 5.3. The use of the classification is somewhat unsatisfactory, and suggests that we should seek a choice of class $\mathcal{FG}$ that is appropriate even when we do not have such a classification. One possibility is to consider all R-modules M which are small as Q-modules, for some map $Q \longrightarrow R$ of algebras from a regular ring Q so that R is small as a Q-module.

5.B. **Classification of regular spaces.** In the rational context we can give a complete classification of regular spaces.

Theorem 5.4. *A simply connected rational space X of finite type is g-regular if and only if $\pi_*(X)$ is even and finite dimensional. It is therefore equivalent to the Eilenberg-MacLane space $K(\pi_*(X))$, and has polynomial cohomology* $\mathrm{Symm}(\pi_*(X))$.

Proof: Since ΩX is a product of Eilenberg-MacLane spaces, we need only remark that odd Eilenberg-MacLane spaces are spheres, whereas even Eilenberg-MacLane spaces are infinite dimensional. $\square$

Proposition 5.5. *If X is g-regular and $H^*(M)$ is finitely generated over $H^*(X)$ then M is small.*

Proof: Suppose $H^*(M)$ is a finitely generated $H^*(X)$-module. Since $H^*(X)$ is a polynomial ring on even degree generators, there is a finite resolution by finitely generated free modules

$$0 \longrightarrow P_r \xrightarrow{d_r} P_{r-1} \xrightarrow{d_{r-1}} \cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} H^*(M) \longrightarrow 0.$$

We proceed to realize this in the usual way. To start with we realize the free modules $P_i = (H^*(X))^{\oplus n}$ by the $C^*(X)$-modules $\mathbb{P}_i = (C^*(X))^{\oplus n}$. Now take $M = M_0$ and realize

the algebraic resolution by constructing a diagram

$$\begin{tikzcd}
\mathbb{P}_0 \arrow[d] & \Sigma\mathbb{P}_1 \arrow[d] & & \Sigma^r\mathbb{P}_r \arrow[d] & \\
M_0 \arrow[r] & M_1 \arrow[r] & \cdots \arrow[r] & M_r \arrow[r] & M_{r+1} \simeq 0
\end{tikzcd}$$

in which the sequences $\Sigma^i\mathbb{P}_i \longrightarrow M_i \longrightarrow M_{i+1}$ are cofibre sequences, $H^*(\Sigma^{-i}M_i) = \ker(d_{i-1})$ and $\Sigma^i\mathbb{P}_i \longrightarrow M_i$ realizes the map in the algebraic resolution. Reversing the process, we see that $M_r, M_{r-1}, \ldots, M_1$ and $M_0 = M$ are finitely built from $C^*(X)$ and therefore small as required. □

This establishes the equivalence of the two definitions of regularity.

Corollary 5.6. *A space is g-regular if and only if it is h-regular.* □

5.C. **Some small objects.** It is useful to identify some modules that are small rather generally.

Lemma 5.7. *If $f : Y \longrightarrow X$ is a map with homotopy fibre $F(f)$ so that $H_*(F(f))$ is finite dimensional, then $C^*(Y)$ is small in $\mathbf{D}(C^*(X))$.*

Proof: By hypothesis, $\mathbb{Q}$ finitely builds $C_*(F(f))$ as a $C_*(\Omega X)$-module. Applying $\mathrm{Hom}_{C_*(\Omega X)}(\mathbb{Q}, \cdot)$, we deduce from the Eilenberg-Moore spectral sequence that $C^*(X)$ finitely builds $C^*(Y)$. In symbols,

$$\mathbb{Q} \models_{C_*(\Omega X)} C_*(F(f))$$

and hence

$$C^*(X) \simeq \mathrm{Hom}_{C_*(\Omega X)}(\mathbb{Q}, \mathbb{Q}) \models \mathrm{Hom}_{C_*(\Omega X)}(\mathbb{Q}, C_*(F(f)) \simeq C^*(Y).$$

□

Lemma 5.8. *If X is g-regular, and $Y \longrightarrow X$ is a map with $C_*(F(f))$ finitely built from $C_*(\Omega X)$ then $C^*(Y)$ is small.*

Proof: Suppose X is g-regular, so that $H_*(\Omega X)$ is finite dimensional. Thus $\mathbb{Q}$ finitely builds $C_*(\Omega X)$. It follows that if $C_*(\Omega X)$ finitely builds $C_*(F(f))$ then $\mathbb{Q}$ finitely builds $C_*(F(f))$ and we may apply the argument of Lemma 5.7. □

6. The centre of a triangulated category.

It will be useful to recall certain constructions before turning to complete intersections.

6.A. **Universal Koszul complexes.** To start with we suppose given a triangulated category $\mathcal{T}$. The *centre* $Z\mathcal{T}$ of $\mathcal{T}$ is defined to be the graded ring of graded endomorphisms of the identity functor.

Given $\chi \in Z\mathcal{T}$ of degree a, for any object X, we may form the mapping cone X/χ of $\chi : \Sigma^a X \longrightarrow X$. This is well defined up to non-unique equivalence. Indeed, given a map $f : X \longrightarrow Y$, the axioms of a triangulated category give a map $f : X/\chi \longrightarrow Y/\chi$ consistent with the defining triangles, but this is not usually unique or compatible with composition.

Now given a sequence of elements $\chi_1, \chi_2, \ldots, \chi_n$ we may iterate this construction, and form

$$K(X;\chi) := X/\chi_1/\chi_2/\cdots/\chi_n,$$

which we refer to as the universal Koszul complex of the sequence. Once again, up to equivalence $K(X;\chi)$ depends only on the sequence, and is independent of the order of the elements χ_i.

6.B. **Bimodules and the centre.** Bimodules provide a useful source of elements of $Z\mathbf{D}(R)$. Indeed, if R is a flat l-algebra, and if $X \longrightarrow Y$ is a map of R-bimodules over l (which is to say, of modules over $R^e = R \otimes_l R$), then for any R-module M we obtain a map

$$X \otimes_R M \longrightarrow Y \otimes_R M$$

of R-modules, natural in M.

It is sometimes convenient to package this in terms of the Hochschild cohomology ring

$$HH^*(R|l) = \mathrm{Ext}^*_{R^e}(R, R).$$

If $l = \mathbb{Z}$, it is usual to omit it from the notation. Now a codegree d element of this cohomology ring can be viewed as a map $R \longrightarrow \Sigma^d R$ in the category of (R,R)-bimodules, so that taking $X = Y = R$ above, we obtain a ring homomorphism

$$HH^*(R|l) \longrightarrow Z\mathbf{D}(R).$$

If R is an l-algebra which is not flat, $R^e = R \otimes_l R$ is taken in the derived sense, and similarly for $HH^*(R|l)$.

Given maps $l \longrightarrow Q \longrightarrow R$, we obtain a map $R \otimes_l R \longrightarrow R \otimes_Q R$ and hence a ring map $HH^*(R|Q) \longrightarrow HH^*(R|l)$. In particular, we have maps

$$R = HH^*(R|R) \longrightarrow HH^*(R|Q) \longrightarrow HH^*(R|\mathbb{Z}) = HH^*(R).$$

If $R = C^*(X)$, we may always take $l = \mathbb{Q} = C^*(pt)$, so that a bimodule is a module over $R^e = C^*(X \times X)$, but it is usually more appropriate to work over $Q = C^*(K)$ where we have a fibration $X \longrightarrow K$. In that case a bimodule over Q is a module over $R^e = C^*(X \times_K X)$.

6.C. **Hochschild cohomology transcended.** It seems natural to relax the role of Hochschild cohomology. For us it is really just a tool for building bimodules from R. We will suppose given a map $Q \longrightarrow R$ so that Q is regular and R is small over Q. This ensures that $\mathcal{FG}$ as defined in Section 5 coincides with the R-modules which are small over Q.

Now, if X is any R-bimodule finitely built from R, we can apply $\otimes_R M$ to deduce $X \otimes_R M$ is finitely built from $M = R \otimes_R M$:

$$R \models_{R^e} X \text{ implies } M = R \otimes_R M \models_R X \otimes_R M.$$

The important case for us is when X is a small R^e-module.

Lemma 6.1. *If X is a small R^e-module and $M \in \mathcal{FG}$, then $X \otimes_R M$ is a small R-module.*

Proof: It suffices to consider the case $X = R^e$. We then have

$$X \otimes_R M = R \otimes_Q R \otimes_R M = R \otimes_Q M.$$

By Proposition 5.5, M is small as a Q-module, so

$$R = R \otimes_Q Q \models R \otimes_Q M$$

as required. $\square$

This comes close to saying that if R is virtually small as an R^e-module then every $M \in \mathcal{FG}$ is virtually small as an R-module. The only obstacle is the need to show $X \otimes_R M$ is non-trivial; in the context we need it, the non-zero degree of the maps constructing X will make it clear.

7. Complete intersection rings and spaces.

We will give definitions of complete intersections as in the regular case. For commutative Noetherian rings these were shown to be equivalent in [9]. We will show they are equivalent for rational spaces.

7.A. **The definition.** In commutative algebra there are three styles for a definition of a complete intersection ring: ideal theoretic, in terms of the growth of the Ext algebra and a derived version.

Definition 7.1. (i) A local Noetherian ring R is a *complete intersection (ci)* ring if $R = Q/(f_1, f_2, \ldots, f_c)$ for some regular ring Q and some regular sequence $f_1, f_2, \ldots, f_c$. The minimum such c (over all Q and regular sequences) is called the *codimension* of R.

(ii) A local Noetherian ring R is *gci* if $\mathrm{Ext}^*_R(k, k)$ has polynomial growth. The *g-codimension* of R is one more than the degree of the growth.

(iii) A local Noetherian ring R is *zci* [9] if there are elements $z_1, z_2, \ldots z_c \in Z\mathbf{D}(R)$ of non-zero degree so that $M/z_1/z_2/\cdots/z_c$ is small for all finitely generated modules M. The minimum such c is called the *z-codimension* of R.

The zci condition implies that every finitely generated module finitely builds a small complex in a prescribed manner using elements in $Z\mathbf{D}(R)$. We can relax this by demanding only that each step in the building of the small complex is the cone of an *endomorphism* of the previous step. This is the essence of the next definition.

(iv) A local Noetherian ring R is *eci* if there is a regular ring Q, a map $Q \longrightarrow R$ and homotopy cofibration sequences of R^e-modules, where $R^e = R \otimes_Q R$,

$$R = M_0 \xrightarrow{g_1} \Sigma^{n_1} M_0 \to M_1, \ \ldots \ , \ M_{c-1} \xrightarrow{g_c} \Sigma^{n_c} M_{c-1} \to M_c$$

such that M_c is small as an R^e-module and the degree of each g_i is not zero.

Two variations are also useful.

(v) A local Noetherian ring R is said to be *bci* if there is a regular ring Q and map $Q \longrightarrow R$ so that R is virtually small as an R^e-module, where $R^e = R \otimes_Q R$.

(vi) If R is a commutative ring or CDGA, it is said to be a *quasi-complete intersection (qci)* [14] if every finitely generated object is virtually small.

Theorem 7.2. [9] *For a local Noetherian ring the conditions ci, gci and zci are all equivalent, and the corresponding codimensions are equal. These conditions imply the eci, bci and qci conditions.*

It is a result of Gulliksen that if R is ci of codimension c, one may construct a resolution of any finitely generated module growing like a polynomial of degree $c-1$. A suitable construction of this resolution shows that R is zci. Considering the module k shows that the ring $\mathrm{Ext}^*_R(k,k)$ has polynomial growth. Perhaps the most striking result about ci rings is the theorem of Gulliksen [20] which states that this characterises ci rings so that the ci and gci conditions are equivalent for local rings.

Remark 7.3. In commutative algebra, Avramov [3] proved Quillen's conjectured characterization of complete intersections by the fact that the André-Quillen cohomology is bounded. When k is of characteristic 0, the DG André-Quillen cohomology of $C^*(X)$ gives the dual homotopy groups of X, so the counterpart of Avramov's characterization is the gci condition.

On the other hand in positive characteristic, results of Mandell [24] show that the topological André-Quillen cohomology of $C^*(X)$ vanishes quite generally, so this does not give an appropriate counterpart of the ci condition.

7.B. **Definitions for spaces.** Adapting the above definitions for spaces is straightforward.

Definition 7.4. (i) A space X is *spherically ci (sci)* if it is formed from a regular space KV using a finite number of spherical fibrations. More precisely, we require that there is a regular space $X_0 = KV$ with V even and finite dimensional, and fibrations

$$S^{n_1} \longrightarrow X_1 \longrightarrow X_0 = KV, S^{n_2} \longrightarrow X_2 \longrightarrow X_1, \ldots, S^{n_c} \longrightarrow X_c \longrightarrow X_{c-1}$$

with $X = X_c$. The least such c is called the *s-codimension* of X.

(ii) A space X is a *gci* space if $H^*(X)$ is Noetherian and $H_*(\Omega X)$ has polynomial growth. The *g-codimension* of X is one more than the degree of growth.

(iii) A space X is a *zci* space if $H^*(X)$ is Noetherian and there are elements $z_1, z_2, \ldots, z_c \in Z\mathbf{D}(C^*(X))$ of non-zero degree so that $C^*(Y)/z_1/z_2/\cdots/z_c$ is small for all $C^*(Y) \in \mathcal{FG}$.

(iv) A space X is an *eci* space if $H^*(X)$ is Noetherian, there is a regular space K and fibration $X \longrightarrow K$ with $C^*(X)$ small over $C^*(K)$ and there are homotopy cofibration sequences of $C^*(X \times_K X)$-modules,

$$C^*(X) = M_0 \xrightarrow{g_1} \Sigma^{n_1} M_0 \to M_1, \ \ldots \ , \ M_{c-1} \xrightarrow{g_c} \Sigma^{n_c} M_{c-1} \to M_c$$

such that M_c is small as an $C^*(X \times_K X)$-module and the degree of each g_i is not zero.

(v) We say X is *bci* space if $H^*(X)$ is Noetherian and $C^*(X)$ is virtually small as a $C^*(X \times_K X)$-module for some regular space K and fibration $X \longrightarrow K$ with $C^*(X)$ small over $C^*(K)$.

(vi) We say X is *qci* space if $H^*(X)$ is Noetherian and each $C^*(Y) \in \mathcal{FG}$ is virtually small.

The main result of this paper is as follows.

Theorem 7.5. *For a rational space X the sci, eci and gci conditions are equivalent. If in addition X is pure, then the conditions above are equivalent to the zci condition.*

We will establish the implications

$$sci \overset{A}{\Rightarrow} eci \overset{B}{\Rightarrow} gci \overset{C}{\Rightarrow} sci.$$

We establish A in Section 10, B in Section 9, and C in Section 12. The first two implications are fairly straightforward in the sense that they can also be proved in the non-rational context [10]. The implication C takes a growth condition and gives a structure theorem, and could be viewed as the main result of the present paper. In Section 11 we show that a pure sci space is zci, while in Section 9 we show that the zci condition implies gci.

Remark 7.6. (i) If X is elliptic then $H^*(X)$ and $\pi_*(\Omega X)$ are both finite dimensional, so it is clear that every elliptic space is gci.

(ii) It is also clear that zci implies qci, and that if the natural transformations giving the zci condition come from Hochschild cohomology then this implies eci, and eci clearly implies bci.

7.C. **Hypersurface rings.** A hypersurface is a complete intersection of codimension 1. The first four definitions adapt to define hypersurfaces, g-hypersurfaces, z-hypersurfaces and e-hypersurfaces. The notion of g-hypersurface (i.e., the dimension of the groups $\mathrm{Ext}^i_R(k,k)$ is bounded) may be strengthened to the notion of p-hypersurface where we require that they are eventually periodic, given by multiplication with an element of the ring. All five of these conditions are equivalent by results of Avramov.

One possible formulation of b-hypersurface would be to require that the R builds a small R^e-module in one step (or equivalently, that R is a z-hypersurface but z arises from $HH^*(R)$). Both these definitions are equivalent to being an e-hypersurface.

Finally, we may say that R is a q-hypersurface if every finitely generated module M has a self map with non-trivial small mapping cone.

7.D. **Hypersurface spaces.** All six of these conditions have obvious formulations for spaces. A space X is an *s-hypersurface* if there is a fibration

$$S^n \longrightarrow X \longrightarrow KV$$

with V even and finite dimensional. It is a *z-hypersurface* if there is an element z of non-zero degree in $Z\mathbf{D}(C^*(X))$ so that, for any M in $\mathcal{FG}$, the mapping cone of $z : M \longrightarrow M$ is small. It is a *g-hypersurface* if the dimensions of $H_i(\Omega X)$ are bounded, and a *p-hypersurace* if they are eventually periodic given by multiplication by an element of the ring.

The space X is an *e-hypersurface* if $C^*(X)$ builds a small $C^*(X \times_K X)$-module in one step for a regular space K. Finally, X is a *q-hypersurface* if every finitely module $C^*(Y)$ in $\mathcal{FG}$ has a self map with non-trivial small mapping cone.

8. Standard form for sci spaces

We are eventually going to show that the sci, gci and eci conditions are equivalent for rational spaces. Of the conditions, the easiest to get a grip on is the sci condition, and it seems worthwhile to begin by anchoring it in reality by giving a structure theorem. In the rational context, we may put sci spaces into a standard form.

Theorem 8.1. *A space X is sci if and only if there exists a fibration sequence*

$$F \to X \to KV,$$

where KV is a regular space and $\pi_*(F)$ is finite dimensional and entirely in odd degrees; in this case

$$\mathrm{codim}(X) = \dim_{\mathbb{Q}}(\pi_*(F)) = \dim_{\mathbb{Q}}(\pi_{odd}(X)).$$

Before proceeding it is useful to note that all the spherical fibrations in the definition of an sci space may be taken to be odd.

Lemma 8.2. *If X can be formed from B with an even spherical fibration $S^{2m} \longrightarrow X \longrightarrow B$, then it can be formed from $B \times K(\mathbb{Q}, 2m)$ by an odd spherical fibration*

$$S^{4m-1} \longrightarrow X \longrightarrow B \times K(\mathbb{Q}, 2m).$$

Accordingly, an sci space of codimension c may be constructed in c steps from a regular space using only odd dimensional spherical fibrations.

Proof: If $C^*(X) = C^*(B) \rtimes \Lambda(x_{2m}, y_{4m-1})$, then if $dy = x^2 + ax + b$ we may change basis by taking $x' = x + a/2$ and find $dx' = 0, dy = (x')^2 + z$, where $z = b - a^2/4 \in C^*(B)$. Adjoining x' to the model of B, we get the base of the required fibration. □

Proof of Theorem 8.1: If X is sci, by Lemma 8.2 we may use only odd spheres in the fibres. Now the composite function $X \longrightarrow KV$ has fibre with only odd dimensional homotopy, giving a fibration of the stated form.

We prove the converse statement by induction on the dimension of the odd homotopy. The result is trivial if the homotopy is entirely even. Suppose then that X lies in a fibration

$$F \longrightarrow X \longrightarrow KV$$

and that $x \in \pi_m(F)$ is an element of highest degree. Construct a fibration

$$S^m \longrightarrow F \longrightarrow F'$$

by killing x, so that $\dim_{\mathbb{Q}}(\pi_*(F')) = \dim_{\mathbb{Q}}(\pi_*(F)) - 1$. Thus, we may choose models so that $C^*(F) = C^*(F') \rtimes C^*(S^m)$, and

$$C^*(X) = C^*(KV) \rtimes [C^*(F') \rtimes C^*(S^m)].$$

Let X' be modelled by the subalgebra generated by $C^*(KV)$ and $C^*(F')$. This gives fibrations

$$S^m \longrightarrow X \longrightarrow X' \text{ and } F' \longrightarrow X' \longrightarrow KV.$$

By induction X' is sci, so that X is sci as required. The codimension is obviously bounded below by $\dim_{\mathbb{Q}}(\pi_{odd}(X))$, and we have described a procedure achieving this bound. □

The following rearrangement result will be useful later.

Corollary 8.3. *If X occurs in a fibration*

$$X' \longrightarrow X \longrightarrow KV$$

with X' sci of codimension c, then X is itself sci of codimension c.

Proof: By Theorem 8.1, X' has a model of the form $X' = KV' \rtimes F'$ with $\pi_*(F')$ finite dimensional and in odd degrees and $X = KV \rtimes X'$ with both V and V' even and finite dimensional. By parity there can be no differential from KV to KV', so

$$X = KV' \rtimes (KV \rtimes F') \simeq (KV' \rtimes KV) \rtimes F'.$$

Since any fibration with base KV' and fibre KV is a product, we obtain a fibration

$$F' \longrightarrow X \longrightarrow K(V \oplus V').$$

By Theorem 8.1 again we deduce X is sci of codimension c. □

In terms of rational models we can restate the sci condition very simply. The result is immediate from Theorem 8.1 by taking a Sullivan model of the fibration.

Corollary 8.4. *A space X is sci if and only if X has a cochain algebra model $(\Lambda V, d)$ where $d(V^{even}) = 0$.* □

9. Growth conditions.

In this section we prove perhaps the simplest implication between the ci conditions: for simply connected rational spaces of finite type, eci (and also zci) implies gci.

9.A. **Polynomial growth.** Throughout algebra and topology it is common to use the rate of growth of homology groups as a measurement of complexity. We will be working over $H^*(X)$, so it is natural to assume that our modules M are *locally finite* in the sense that $H^*(M)$ is cohomologically bounded below and $\dim_{\mathbb{Q}}(H^i(M))$ is finite for all i.

Definition 9.1. We say that a locally finite module M has *polynomial growth of degree* $\leq d$, and write $\mathrm{growth}(M) \leq d$, if there is a polynomial $p(x)$ of degree d with

$$\dim_{\mathbb{Q}}(H^n(M)) \leq p(n)$$

for all $n >> 0$.

Remark 9.2. (i) In commutative algebra the usual terminology is that a module of growth d has *complexity* $d + 1$.

(ii) Note that a complex with bounded homology has growth ≤ -1. For complexes with growth $\leq d$ with $d \geq 0$, by adding a constant to the polynomial, we may insist that the bound applies for all $n \geq 0$.

9.B. **Mapping cones reduce degree by one.** We use the following estimate on growth.

Lemma 9.3. *Given cohomologically bounded below locally finite modules M and N in a triangle*

$$\Sigma_n M \xrightarrow{\chi} M \longrightarrow N$$

with $n \neq 0$, then

$$\mathrm{growth}(M) \leq \mathrm{growth}(N) + 1.$$

Proof: The homology long exact sequence of the triangle includes

$$\cdots \longrightarrow H^{i-n}(M) \xrightarrow{\chi} H^i(M) \longrightarrow H^i(N) \longrightarrow \cdots .$$

This shows

$$\dim_{\mathbb{Q}}(H^i(M)) \leq \dim_{\mathbb{Q}}(H^i(N)) + \dim_{\mathbb{Q}}(\chi H^{i-n}(M)).$$

Iterating s times, we find

$$\dim_{\mathbb{Q}}(H^i(M)) \leq \dim_{\mathbb{Q}}(H^i(N)) + \dim_{\mathbb{Q}}(H^{i-n}(N)) + \cdots$$
$$\cdots + \dim_{\mathbb{Q}}(H^{i-(s-1)n}(N)) + \dim_{\mathbb{Q}}(\chi^s H^{i-sn}(M)).$$

To obtain growth estimates, it is convenient to collect the dimensions of the homogeneous parts into the Hilbert series $h_M(t) = \sum_n \dim_{\mathbb{Q}}(H^i(M))t^i$. An inequality between such formal series means that it holds between all coefficients.

First suppose that $n > 0$. Since $H^*(M)$ is bounded below, if $h_M(t)$ is the Hilbert series of $H^*(M)$ then we have

$$h_M(t) \leq h_N(t)(1 + t^n + t^{2n} + \cdots) = \frac{h_N(t)}{1 - t^n},$$

giving the required growth estimate.

If $n = -n' < 0$ we rearrange to obtain

$$N' \longrightarrow M \longrightarrow \Sigma_{n'} M$$

where $N' = \Sigma_{n'-1} N$ and argue precisely similarly. □

9.C. **Growth of eci spaces.** The implication we require is now straightforward.

Theorem 9.4. *If X is eci then it is also gci, and if X has e-codimension c it has g-codimension $\leq c$.*

Proof: It is sufficient to show $C^*(\Omega X) \simeq \mathbb{Q} \otimes_{C^*(X)} \mathbb{Q}$ has polynomial growth.

By hypothesis there is an appropriate regular space K and self maps

$$\gamma_1 : M_0 \to \Sigma_{|\gamma_1|} M_0,\ \gamma_2 : M_1 \to \Sigma_{|\gamma_1|} M_1,\ \ldots\ ,\ \gamma_c : M_{c-1} \to \Sigma_{|\gamma_c|} M_{c-1}$$

of non-zero degree in $\mathbf{D}(C^*(X \times_K X))$, so that M_i is the cone of γ_i and M_c, which is the cone of γ_c, is small. Thus, applying $\mathbb{Q} \otimes_{C^*(X)} (\cdot)$ to M_c we obtain a complex with growth ≤ -1. By the lemma if we apply $\mathbb{Q} \otimes_{C^*(X)} (\cdot)$ to M_{c-1} we obtain a complex of growth ≤ 0. Doing this repeatedly, we deduce that when we apply $\mathbb{Q} \otimes_{C^*(X)} (\cdot)$ to $\mathbb{Q}$ itself we obtain a complex with growth $\leq c - 1$ as required. □

The proof above, with minor changes, also yields the following Theorem.

Theorem 9.5. *If X is zci then it is also gci, and if X has z-codimension c it has g-codimension $\leq c$.*

10. SCI SPACES ARE ECI SPACES

In this section we show that sci spaces (defined by a particular construction) have a periodic module theory in the sense that they are eci. This may not be too surprising, but the particular way in which bimodules and fibrations are used may be of some interest.

Theorem 10.1. *If X is an sci space of codimension c, then it is eci of codimension c.*

Remark 10.2. The construction will show that all the maps building the small bimodule are of positive degree, so that Lemma 6.1 shows that if X is sci then all $C^*(X)$-modules in $\mathcal{FG}$ are virtually small.

We will upgrade the conclusion to show that if X is a pure sci space then X is zci in Section 11.

10.A. **Fibration lemmas.** We will repeatedly use two elementary lemmas. The first is very well known.

Lemma 10.3. *If*

$$F \longrightarrow E \xrightarrow{p} B$$

is a fibration with a section s, then there is a fibration

$$\Omega F \longrightarrow B \xrightarrow{s} E.$$

Proof: We start from the square

$$\begin{tikzcd} B \arrow[r] \arrow[d, "="] & E \arrow[d, "p"] \\ B \arrow[r, "="] & B \end{tikzcd}$$

and take iterated fibres. □

The second lemma is a Third Isomorphism Theorem for fibrations.

Lemma 10.4. *Given fibrations $Y \longrightarrow B \longrightarrow C$, if $F = \mathrm{fibre}(B \longrightarrow C)$ there is a fibration*

$$\Omega F \longrightarrow Y \times_B Y \longrightarrow Y \times_C Y.$$

Proof: We start from the cube

$$\begin{tikzcd}[row sep=small, column sep=small] & Y \times_C Y \arrow[rr] \arrow[dd] & & Y \arrow[dd] \\ Y \times_B Y \arrow[ur] \arrow[rr] \arrow[dd] & & Y \arrow[ur] \arrow[dd] & \\ & Y \arrow[rr] & & C \\ Y \arrow[ur] \arrow[rr] & & B \arrow[ur] & \end{tikzcd}$$

and take iterated fibres. □

10.B. **Building bimodules.** It is worth isolating the process that we use repeatedly to build bimodules. Abstracted from its context, the proof is extremely simple. The strength of the result is that the cofibre sequence is one of $C^*(Y)$-modules.

Proposition 10.5. *Suppose given a fibration*

$$\Omega S^m \longrightarrow X \xrightarrow{f} Y.$$

(i) If m is odd, then there is a cofibre sequence of $C^*(Y)$-modules

$$\Sigma_{m-1} C^*(X) \longleftarrow C^*(X) \xleftarrow{f^*} C^*(Y).$$

(ii) If m is even, then there is a cofibre sequence of $C^*(Y)$-modules

$$\Sigma_{2m-2} C^*(X) \longleftarrow C^*(X) \longleftarrow F$$

with F small. More precisely, F is built from two copies of $C^*(Y)$ in the sense that there is a cofibre sequence

$$\Sigma_{m-1}C^*(Y) \longleftarrow F \longleftarrow C^*(Y)$$

of $C^*(Y)$-modules.

Proof: We take a relative Sullivan model for f^*. This has the form $C^*(Y) \rtimes C^*(\Omega S^m)$.

If m is odd, the relative Sullivan model is of the form $C^*(X) \simeq C^*(Y) \rtimes \Lambda(z_{m-1})$, where z is a polynomial generator. The quotient of this by the DG-$C^*(Y)$-submodule $C^*(Y) \cdot z^0$ is again a $C^*(Y)$-module, and the composite

$$\Sigma_{m-1}C^*(X) \xrightarrow{z} C^*(X) \longrightarrow C^*(X)/C^*(Y)$$

is an isomorphism as required.

If m is even, the relative Sullivan model is of the form

$$C^*(X) \simeq C^*(Y) \rtimes \Lambda(z_{m-1}, t_{2m-2}).$$

This time t is a polynomial generator, and z is an exterior generator. Accordingly we let F be the DG-$C^*(Y)$-submodule generated by t^0 and z, so that $C^*(Y)/F$ is again a $C^*(Y)$-module. It is isomorphic to $\Sigma_{2m-2}C^*(X)$ in the sense that the composite

$$\Sigma_{2m-2}C^*(X) \xrightarrow{t} C^*(X) \longrightarrow C^*(X)/F$$

is an isomorphism. $\square$

Remark 10.6. As an example, we observe that this shows that an s-hypersurface is a z-hypersurface. Indeed, by hypothesis, we have a fibration $S^m \longrightarrow X \longrightarrow KV$, and hence by pullback a fibration

$$S^m \longrightarrow X \times_{KV} X \longrightarrow X$$

with a section $\Delta : X \longrightarrow X \times_{KV} X$. By Lemma 10.3, we obtain a fibration

$$\Omega S^m \longrightarrow X \xrightarrow{\Delta} X \times_{KV} X$$

so we may apply Proposition 10.5 with $Y = X \times_{KV} X$, noting that a $C^*(Y)$-module is then a $C^*(X)$-bimodule. If m is odd we then get a cofibre sequence

$$\Sigma_{m-1}C^*(X) \xleftarrow{\tilde{\chi}} C^*(X) \longleftarrow C^*(X \times_{KV} X)$$

of bimodules. As in Subsection 6.C note that $\tilde{\chi}$ gives an element of $Z\mathbf{D}(C^*(X))$ by tensoring down, in the sense that for any $C^*(X)$-module M we apply $M \otimes_{C^*(X)} (\cdot)$ to get a cofibre sequence

$$\Sigma_{m-1}M \xleftarrow{\chi} M \longleftarrow C^*(X) \otimes_{C^*(KV)} M.$$

If M is finitely generated, then it is small as a $C^*(KV)$-module by Proposition 5.5 showing that the fibre of χ is small as required.

The argument when m is even is precisely similar.

10.C. **The proof.** We now have the necessary ingredients for proving Theorem 10.1.

We suppose X is sci of codimension c, so that we may form $X = X_c$ in c steps from $X_0 = KV$ using fibrations

$$S^{n_i} \longrightarrow X_i \longrightarrow X_{i-1}.$$

It will simplify the argument to assume all the spheres are odd dimensional, as we may do by Lemma 8.2.

We must show that $C^*(X)$ builds $C^*(X \times_{KV} X)$ as a bimodule (i.e., as a $C^*(X \times_{KV} X)$-module) using s cofibre sequences. It is convenient to write $X_i^e = X \times_{X_i} X$, and $X^e = X_0^e$, so that we want to work with $C^*(X^e)$-modules. However, since we have maps

$$X = X_s \longrightarrow X_{s-1} \longrightarrow \cdots \longrightarrow X_0 = KV,$$

we have maps

$$X_s = X_s^e \longrightarrow X_{s-1}^e \longrightarrow \cdots \longrightarrow X_0^e = X^e,$$

so we may view $C^*(X_i^e)$-modules as $C^*(X_0^e)$-modules by restriction.

We are ready to apply our fibration lemmas.

Pulling back the fibration along $X_i \longrightarrow X_{i-1}$ we obtain a fibration

$$S^{n_i} \longrightarrow X_i \times_{X_{i-1}} X_i \xrightarrow{\pi_1} X_i$$

with a section given by the diagonal Δ. Applying Lemma 10.3 we obtain a fibration

$$\Omega S^{n_i} \longrightarrow X_i \xrightarrow{\Delta} X_i \times_{X_{i-1}} X_i.$$

Similarly, applying Lemma 10.4 to $X_s \longrightarrow X_i \longrightarrow X_{i-1}$ where $s \geq i$, we obtain a fibration

$$\Omega S^{n_i} \longrightarrow X_i^e \longrightarrow X_{i-1}^e.$$

Now using the first of these, Proposition 10.5 gives a cofibration

$$\Sigma_{n_s} C^*(X) \longleftarrow C^*(X) \longleftarrow C^*(X_{s-1}^e),$$

of $C^*(X_{s-1}^e)$-modules, which we view as a cofibration of $C^*(X^e)$-modules by pullback. Successive fibrations give

$$\Sigma_{n_i} C^*(X_i^e) \longleftarrow C^*(X_i^e) \longleftarrow C^*(X_{i-1}^e),$$

until we reach

$$\Sigma_{n_1} C^*(X_1^e) \longleftarrow C^*(X_1^e) \longleftarrow C^*(X^e),$$

so that

$$C^*(X) = C^*(X_s^e) \models C^*(X_0^e) = C^*(X^e)$$

as required. □

11. Hochschild cohomology and pure Sullivan algebras

In this section we calculate the Hochschild cohomology of a pure sci space X and upgrade the conclusion of Section 10 to give the required conclusion that any pure sci space is also zci.

Theorem 11.1. *If X is a pure sci space of codimension c, then it is zci of codimension c.*

In view of Theorem 10.1 and Remark 10.2 we need only show that the maps of bimodules used in the constructions of Section 10 all lift to elements of $Z\mathbf{D}(R)$. Specifically, this is Corollary 11.7.

11.A. **Hochschild cohomology.** It is convenient to adapt the algebraic notation for Hochschild cohomology to cochain algebras.

Notation 11.2. If $X \to Y$ is a fibration of spaces, set

$$HH^*(X|Y) := \mathrm{Ext}^*_{C^*(X\times_Y X)}(C^*(X), C^*(X)).$$

Note that we consider $C^*(X)$ as a $C^*(X \times_Y X)$-module via the diagonal map $X \to X \times_Y X$.

We will be applying this to sci spaces, and use the notation of Section 10: $X = X_s$ is an sci space of codimension s, so that for $1 \leq i \leq s$ we have fibrations $S^{n_i} \to X_i \to X_{i-1}$ with n_i odd and $X_0 = KV$, where V is finite dimensional and even.

Theorem 11.3. *If X is a pure sci space as above, the Hochschild cohomology is given by*

$$HH^*(X|KV) = H^*(X)[[\zeta_1, ..., \zeta_s]],$$

where the degree of ζ_i is $n_i - 1$.

Remark 11.4. (i) The completion involved in forming the power series ring is homogeneous, so that if X is finite dimensional the ring is a polynomial ring. Otherwise the formula is to be interpreted as formed by successive adjunction of the variables in the stated order.

(ii) One would expect it to follow from Theorem 11.3 that the construction of Section 10 can be upgraded to show X is zci. In any case, this upgrading of the construction is an ingredient in the Hochschild cohomology calculation.

The theorem evidently follows by repeated application of the following general result about fibrations with fibre an odd sphere.

Proposition 11.5. *Suppose given a fibration sequence $S^{2n+1} \longrightarrow Y \longrightarrow Z$, and a fibration $X \longrightarrow Y$. Assuming (i) $\pi_*(Y) \longrightarrow \pi_*(Z)$ is surjective, (ii) $\pi_*(X) \longrightarrow \pi_*(Y)$ is surjective, with kernel concentrated in odd degrees, and (iii) $\pi_*(X)$ is finite dimensional and X is a pure space, we have*

$$HH^*(X|Z) \cong HH^*(X|Y)[[\zeta]],$$

where ζ is of degree $2n$.

This will be proved in Subsection 11.F below.

11.B. **Upgrading bimodule maps.** Using the notation from Proposition 11.5, we suppose given a fibration $S^{2n+1} \longrightarrow Y \longrightarrow Z$, and a map $X \longrightarrow Y$. We write $A = C^*(X)$, $B = C^*(Y)$ and $C = C^*(Z)$.

It is shown in Section 10 (see Lemma 10.4 and Proposition 10.5) that there is a cofibre sequence of $A \otimes_C A$-modules

$$A \otimes_C A \to A \otimes_B A \xrightarrow{\varphi} \Sigma_{2n} A \otimes_B A.$$

To obtain an element of $Z\mathbf{D}(R)$ we proceed as follows. For each A-module M, we apply $- \otimes_A M$ to obtain the cofibre sequence

$$A \otimes_C M \to A \otimes_B M \xrightarrow{\varphi \otimes_A M} \Sigma_{2n} A \otimes_B M.$$

To establish the zci condition, we must check it is natural for maps of A-modules.

Proposition 11.6. *There is a morphism* $\zeta : A \to \Sigma_{2n} A$ *of* $A \otimes_C A$*-modules (i.e., an element* $\zeta \in HH^{2n}(A|C)$*) such that*

$$\zeta \otimes_B A \simeq \varphi.$$

We will prove the proposition in Subsection 11.E below. For the present we just observe that it has the desired consequence.

Corollary 11.7. *There is a natural transformation* z *of the identity functor on* A*-modules such that for every* A*-module* M

$$z(A \otimes_B M) \simeq \varphi \otimes_A M.$$

Proof: The natural transformation z that ζ induces on A-modules is given by $z(M) = \zeta \otimes_A M$. We easily verify this has the required property:

$$\begin{aligned} z(A \otimes_B M) &= \zeta \otimes_A (A \otimes_B M) \\ &= \zeta \otimes_B M \\ &= (\zeta \otimes_B A) \otimes_A M \\ &= \varphi \otimes_B M. \end{aligned}$$

□

This completes the proof of Theorem 11.1.

11.C. **Models for spaces.** Using the notation of Proposition 11.5, we work with a fibration sequence $S^{2n+1} \to Y \to Z$ and a fibration $X \longrightarrow Y$, and we let A, B and C be minimal Sullivan models for X, Y and Z respectively. More explicitly, we take models as follows.

(1) $C = (\Lambda W, d)$ with W finite dimensional.
(2) $B = (\Lambda(W \oplus \mathbb{Q}x_{2n+1}), d)$, containing C as a sub-algebra. We denote $W \oplus \mathbb{Q}x$ by V.
(3) $A = (\Lambda(V \oplus U), d)$, containing B as a sub-algebra, with U concentrated in odd degrees and finite dimensional. We also assume that $d(U) \subset \Lambda W$, this is possible because X has a pure Sullivan model.

Let $X^e_Y = X \times_Y X$ and let $X^e_Z = X \times_Z X$. The cochain algebras $A^e_B = A \otimes_B A$ and $A^e_C = A \otimes_C A$ are minimal Sullivan models for X^e_Y and X^e_Z. We can write these cochain algebras explicitly as well:

- $A^e_B = (\Lambda(V \oplus U_l \oplus U_r), d)$ where $U_l = \{u_l | u \in U\}$ and $U_r = \{u_r | u \in U\}$. The differential d is the obvious one satisfying $d(u_l) = d(u_r) = d(u) \in \Lambda W$ and so $u_l - u_r$ is always a cocycle.
- $A^e_C = (\Lambda(W \oplus \mathbb{Q}\{x_l, x_r\} \oplus U_l \oplus U_r), d)$.

Remark 11.8. There are three important morphisms for X^e_Y:

(1) $l : X^e_Y \to X$ which is mapping to the left component,
(2) $r : X^e_Y \to X$ which is mapping to the right component and
(3) $\Delta : X \to X^e_Y$ which is the diagonal.

The algebraic counterparts of these maps are $l : A \to A^e_B$, $r : A \to A^e_B$ and $\Delta : A^e_B \to A$. The morphism l is the map to the left component of $A \otimes_B A = A^e_B$, it is defined by $l(u) = u_l$ for $u \in U$ and $l(v) = v$ for $v \in V$. The description of $r : A \to A^e_B$ is precisely similar. The diagonal map $\Delta : A^e_B \to A$ is defined by $\Delta(u_l) = \Delta(u_r) = u$ and Δ is the identity on

$\mathbb{Q}x$ and W. There are similar maps for X_Z^e and A_C^e. Note that A is a A_C^e-module via the morphism $\Delta : A_C^e \longrightarrow A$.

11.D. **Some useful fibrations.** The following fibrations will be central to the proof.

Lemma 11.9. *There are the following fibration sequences,*

(1) $X_Y^e \xrightarrow{\Delta} X_Z^e \to S(\mathbb{Q}x)$.
(2) $X \xrightarrow{\Delta} X_Y^e \to S(U)$.
(3) $X \xrightarrow{\Delta} X_Z^e \to S(U \oplus \mathbb{Q}x)$.

where the bases are products of odd spheres with the indicated homotopy groups.

Proof: We reformulate the lemma algebraically: there are the following cofibration sequences of Sullivan algebras:

(1) $L_1 = (\Lambda\mathbb{Q}x', 0) \to A_C^e \xrightarrow{\Delta} A_B^e$, where $x' \mapsto x_l - x_r$.
(2) $L_2 = (\Lambda U', 0) \to A_B^e \xrightarrow{\Delta} A$, where $U' \cong U$ and $u' \mapsto u_l - u_r$.
(3) $L_1 \otimes_{\mathbb{Q}} L_2 = (\Lambda(\mathbb{Q}x' \oplus U'), 0) \to A_C^e \xrightarrow{\Delta} A$.

Since the composite $(\Lambda\mathbb{Q}x', 0) \to A_C^e \xrightarrow{\Delta} A_B^e$ is the trivial morphism, there is a natural morphism $\epsilon : A_C^e \otimes_{L_1} \mathbb{Q} \to A_B^e$. It is easy to see that ϵ is an isomorphism on homotopy groups. The proof for the two other cofibration sequences is similar. □

We shall make two uses of these fibrations. The first use is to build relative cofibrant models for our cochain algebras. An A_C^e-cofibrant model for A_B^e is

$$\tilde{A_B^e} = (\Lambda(W \oplus \mathbb{Q}\{x_l, x_r, z_{2n}\} \oplus U_l \oplus U_r), d) \text{ where } dz = x_l - x_r.$$

It is easy to see that the obvious morphism $\tilde{A_B^e} \to A_B^e$ is indeed a weak equivalence. Similarly a cofibrant model for A over $\tilde{A_B^e}$ (and therefore also over A_C^e) is given by the formula

$$\tilde{A} = (\Lambda(W \oplus \mathbb{Q}\{x_l, x_r, z_{2n}\} \oplus U_l \oplus U_r \oplus \tilde{U}), d) \text{ where } \tilde{U} = \Sigma_1 U \text{ and } d(\tilde{u}) = u_l - u_r.$$

The second use of Lemma 11.9 is in defining a strange and useful space T.

Lemma 11.10. *Let T be the homotopy fibre of the map $X_Z^e \to S(U')$.*

(1) *There is a fibration sequence $X \to T \to S(\mathbb{Q}x)$.*
(2) *The following is a homotopy pullback square*

$$\begin{tikzcd} X \arrow[r, "\Delta"] \arrow[d] & X_Y^e \arrow[d] \\ T \arrow[r] & X_Z^e \end{tikzcd}$$

Proof: An A_C^e-cofibrant cochain model for T is given by the formula

$$F = (\Lambda(W \oplus Q\{x_l, x_r\} \oplus U_l \oplus U_r \oplus \tilde{U}), d) \text{ where } \tilde{U} = \Sigma_1 U \text{ and } d(\tilde{u}) = u_l - u_r.$$

Note that $u_l - u_r$ is a cocycle because $d(U) \subset \Lambda W$. From this model the fibration sequence $X \to T \to S(\mathbb{Q}x)$ is evident.

Let X' be the homotopy pullback of the diagram:

$$\begin{tikzcd} & X_Y^e \arrow[d] \\ T \arrow[r] & X_Z^e. \end{tikzcd}$$

A cochain algebra model for X' is $F \otimes_{A_C^e} \tilde{A_B^e} = (\Lambda(W \oplus Q\{x_l, x_r\} \oplus U_l \oplus U_r \oplus \tilde{U} \oplus \mathbb{Q}z_{2n}), d)$ which is clearly isomorphic to $\tilde{A}$. Moreover, the morphism $\tilde{A_B^e} \to F \otimes_{A_C^e} \tilde{A_B^e} \cong \tilde{A}$ is indeed the diagonal Δ. □

11.E. **Lifting the map of bimodules.** We can now prove Proposition 11.6. Recall the cochain model F for T given in the proof of Lemma 11.10. The fibration $X \to T \to S(\mathbb{Q}x)$ induces an exact sequence of F-modules (and therefore also of A_C^e-modules):

$$F \to \tilde{A} \xrightarrow{\zeta} \Sigma_{2n}\tilde{A}.$$

We will require an explicit description of ζ. It is defined by

- $\zeta(fz^q) = fz^{q-1}$ and
- $\zeta(f) = 0$ if f is not divisible by z.

Similarly, the fibration sequence $X_Y^e \to X_Z^e \to S(\mathbb{Q}x)$ gives rise to an exact sequence

$$A_C^e \to \tilde{A_B^e} \xrightarrow{\varphi} \Sigma_{2n}\tilde{A_B^e}$$

of A_C^e-modules. An explicit description of φ is given by

- $\varphi(fz^q) = fz^{q-1}$ and
- $\varphi(f) = 0$ if f is not divisible by z.

As a cofibrant replacement of B over $B \otimes_C B$, we take $\tilde{B} = (\Lambda(W \oplus \mathbb{Q}\{x_l, x_r, z_{2n}\}), d)$ with $dz = x_l - x_r$. We can now prove the following proposition, which is an explicit cochain level version of Proposition 11.6.

Proposition 11.11. *There is a natural equivalence*

$$\varphi \simeq \zeta \otimes_{\tilde{B}} \tilde{A}$$

Proof: We shall define a cochain algebra model $\hat{A}$ for X, which is cofibrant over $\tilde{B}$. Let $\hat{A} = (\Lambda(W \oplus \mathbb{Q}\{x_l, x_r, z_{2n}\} \oplus U_l), d)$ where $dz = x_l - x_r$. The morphism ζ is equivalent to the obvious morphism $\hat{\zeta} : \hat{A} \to \Sigma_{2n}\hat{A}$. It is now easy to see there is an equality of morphisms of A_C^e-modules:

$$\varphi = \hat{\zeta} \otimes_{\tilde{B}} \hat{A}$$

□

Remark 11.12. Proposition 10.5 shows that the fibration $S^{2n+1} \to Y \to Z$ yields an exact sequence:

$$B \otimes_C B \to \tilde{B} \xrightarrow{\psi} \Sigma_{2n}\tilde{B}.$$

It is easy to see that $\varphi = A \otimes_B \psi \otimes_B A$ (note that ψ is a morphism of $B \otimes_C B$-modules, which justifies tensoring over B on the left and right).

11.F. **Proof of Proposition 11.5.** Using our cofibrant models, we have explicit complexes for calculating Hochschild cohomology:

$$HH^*(A|C) = H^*(\mathrm{End}_{A^e_C}(\tilde{A})) \text{ and } HH^*(A|B) = H^*(\mathrm{End}_{\tilde{A^e_B}}(\tilde{A})).$$

In these terms, we may state Proposition 11.5 more explicitly as follows.

Proposition 11.13.

$$H_*(\mathrm{End}_{A^e_C}(\tilde{A})) = H_*(\mathrm{End}_{\tilde{A^e_B}}(\tilde{A}))[[\zeta]]$$

Proof: Let $R = H^*(\mathrm{End}_{A^e_C}(\tilde{A}))$ and let $Q = H^*(\mathrm{End}_{\tilde{A^e_B}}(\tilde{A}))$. Note that both R and Q are graded-commutative, because Hochschild cohomology is always graded-commutative. To prove the proposition we need several ingredients. The first ingredient is a short exact sequence

$$\Sigma^{2n} R \xrightarrow{\zeta} R \to Q$$

of R-modules.

Consider the morphism $F \xrightarrow{p} \tilde{A}$ of Lemma 11.10. Applying the functor $\mathrm{Hom}_{A^e_C}(-, \tilde{A})$ yields a morphism $\mathrm{End}_{A^e_C}(\tilde{A}) \xrightarrow{p^*} \mathrm{Hom}_{A^e_C}(F, \tilde{A})$. Since $\tilde{A}$ is a $\tilde{A^e_B}$-module there is an adjunction:

$$\mathrm{Hom}_{A^e_C}(F, \tilde{A}) \cong \mathrm{Hom}_{\tilde{A^e_B}}(\tilde{A^e_B} \otimes_{A^e_C} F, \tilde{A}) \cong \mathrm{End}_{\tilde{A^e_B}}(\tilde{A})$$

(the isomorphism $\tilde{A^e_B} \otimes_{A^e_C} F \cong \tilde{A}$ is Part 2 of Lemma 11.10). Thus p^* is a map $\mathrm{End}_{A^e_C}(\tilde{A}) \to \mathrm{End}_{\tilde{A^e_B}}(\tilde{A})$. On the other hand, we have the natural multiplicative change of rings map $\iota : \mathrm{End}_{\tilde{A^e_B}}(\tilde{A}) \to \mathrm{End}_{A^e_C}(\tilde{A})$. Using the explicit construction of internal Hom of DG-modules over a CDGA, it is straightforward to verify that p^* is left inverse to ι.

Next, consider the short exact sequence $F \to \tilde{A} \xrightarrow{\zeta} \Sigma_{2n}\tilde{A}$. Applying $\mathrm{Hom}_{A^e_C}(-, \tilde{A})$ to this sequence yields a distinguished triangle of left $\mathrm{End}_{A^e_C}(\tilde{A})$-modules

$$\Sigma^{2n}\mathrm{End}_{A^e_C}(\tilde{A}) \xrightarrow{\zeta^*} \mathrm{End}_{A^e_C}(\tilde{A}) \xrightarrow{p^*} \mathrm{End}_{\tilde{A^e_B}}(\tilde{A}).$$

The morphism $\zeta^* : \Sigma^{2n}\mathrm{End}_{A^e_C}(\tilde{A}) \to \mathrm{End}_{A^e_C}(\tilde{A})$ is just composition with ζ, i.e., right multiplication by $\zeta \in \mathrm{End}_{A^e_C}(\tilde{A})$. This distinguished triangle yields a long exact sequence of homology groups. Since $H_*(p^*)$ is an epimorphism, we have a short exact sequence of graded left R-modules:

$$\Sigma^{2n} R \xrightarrow{\zeta} R \xrightarrow{H_*(p^*)} Q.$$

Note that there are two multiplicative structure on Q: the usual one and the one coming from Q being a quotient of the graded ring R by the ideal (ζ). These structures must coincide, because $H_*(\iota)$ is multiplicative and has a left inverse.

The second ingredient is that the homotopy inverse limit of the tower

$$\mathcal{T} := \left[R \xleftarrow{z} R \xleftarrow{z} \cdots \right]$$

is zero. Because A is zero in negative codegrees, the homotopy colimit $\tilde{A}^\infty$ of the telescope $\tilde{A} \xrightarrow{\zeta} \Sigma_{2n}\tilde{A} \xrightarrow{\zeta} \cdots$ is zero. Applying $\mathrm{Hom}_{A^e_C}(-, \tilde{A})$ to this telescope gives a tower

$$\mathrm{End}_{A^e_C}(\tilde{A}) \xleftarrow{\zeta^*} \Sigma^{2n}\mathrm{End}_{A^e_C}(\tilde{A}) \xleftarrow{\zeta^*} \cdots$$

Its homotopy inverse limit, $\mathrm{Hom}_{A^e_C}(\tilde{A}^\infty, \tilde{A})$ is therefore also zero. By the Milnor exact sequence, $\varprojlim \mathcal{T} = 0$ and $R^1 \varprojlim \mathcal{T} = 0$.

Next, consider the two towers:

(1) $\mathcal{U} = \left[R \xleftarrow{=} R \xleftarrow{=} \cdots\right]$, and

(2) $\mathcal{V} = [Q = R/(\zeta) \leftarrow R/(\zeta^2) \leftarrow R/(\zeta^3) \leftarrow \cdots]$.

There is a short exact sequence of towers $0 \to \mathcal{T} \to \mathcal{U} \to \mathcal{V} \to 0$, given by:

$$\begin{tikzcd}
R \arrow[d, "\zeta"] & R \arrow[l, "\zeta"'] \arrow[d, "\zeta^2"] & R \arrow[l, "\zeta"'] \arrow[d, "\zeta^3"] & \cdots \arrow[l, "\zeta"'] \\
R \arrow[d] & R \arrow[l, "="'] \arrow[d] & R \arrow[l, "="'] \arrow[d] & \cdots \arrow[l, "="'] \\
Q & R/(\zeta^2) \arrow[l] & R/(\zeta^3) \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

The six term exact sequence:

$$0 \to \varprojlim \mathcal{T} \to \varprojlim \mathcal{U} \to \varprojlim \mathcal{V} \to R^1 \varprojlim \mathcal{T} \to R^1 \varprojlim \mathcal{U} \to R^1 \varprojlim \mathcal{V} \to 0$$

shows that R (which is isomorphic to $\varprojlim \mathcal{U}$) is isomorphic to $\varprojlim \mathcal{V}$.

To complete the proof we need to show that $R/(\zeta^n)$ is isomorphic to the truncated polynomial ring $Q[\zeta]/(\zeta^n)$. Observe that $(\zeta^n)/(\zeta^{n+1})$ is isomorphic to Q as an R-module, and that there is a subalgebra $Q[\zeta] \subseteq R$. These facts yield a morphism of short exact sequences:

$$\begin{tikzcd}
Q \arrow[r] \arrow[d, "\cong"] & Q[\zeta]/(\zeta^{n+1}) \arrow[r] \arrow[d] & Q[\zeta]/(\zeta^n) \arrow[d] \\
(\zeta^n)/(\zeta^{n+1}) \arrow[r] & R/(\zeta^{n+1}) \arrow[r] & R/(\zeta^n)
\end{tikzcd}$$

Since $R/(\zeta) \cong Q$ we get an inductive argument showing that $R/(\zeta^n) \cong Q[\zeta]/(\zeta^n)$ as rings.

Therefore

$$R \cong \varprojlim Q[\zeta]/(\zeta^n) = Q[[\zeta]],$$

as required. □

12. Polynomial growth implies spherical extension.

The purpose of the present section is to complete the loop of implications and prove the following theorem.

Theorem 12.1. *If X is a gci space, it is also sci.*

This states that a finiteness condition (Noetherian cohomology and finite homotopy) implies that a space has a particular form (fibration $F \longrightarrow X \longrightarrow KV$, where $\pi_*(F)$ is in odd degrees) and is therefore perhaps the most interesting step.

12.A. **Strategy.** Assume X is gci. By the Milnor-Moore theorem, $\pi_*(\Omega X)$ is finite dimensional, and in particular its even part is finite dimensional.

We argue by induction on $\dim_{\mathbb{Q}}(\pi_*(X))$. The result is trivial if $\pi_*(X) = 0$, and the inductive step will be to remove the top homotopy group and retain the Noetherian condition. Suppose then that the top non-zero homotopy is in degree s and that $0 \neq x \in \pi_s^\vee(X)$.

If $s = 2n-1$ is odd, killing homotopy groups gives a fibration

$$S^{2n-1} \longrightarrow X \longrightarrow X',$$

with $\dim_{\mathbb{Q}}(\pi_*(X')) = \dim_{\mathbb{Q}}(\pi_*(X)) - 1$. In the rational setting, the fibration is principal, so there is also a fibration

$$X \longrightarrow X' \longrightarrow K(\mathbb{Q}, 2n-2),$$

and we may use the Serre spectral sequence to deduce that $H^*(X')$ is Noetherian. Thus X' is gci, and by induction we conclude it is also sci. The fibration displays X as being sci as required.

If $s = 2n$ is even, killing homotopy groups gives a fibration, $K(\mathbb{Q}, 2n) \longrightarrow X \longrightarrow Y$, but this is not of use to us. We will argue, heavily using the fact that $H^*(X)$ is Noetherian, that in fact the element x is in the image of the dual Hurewicz map. Accordingly there is another fibration

$$X' \longrightarrow X \longrightarrow K(\mathbb{Q}, 2n)$$

where $\dim_{\mathbb{Q}}(\pi_*(X')) = \dim_{\mathbb{Q}}(\pi_*(X)) - 1$. Applying the Serre spectral sequence to the fibration

$$S^{2n-1} \longrightarrow X' \longrightarrow X,$$

we see that $H^*(X')$ is Noetherian. By induction we conclude X' is sci, and from Corollary 8.3 it follows that X is sci.

12.B. **The dual Hurewicz map.** In rational homotopy it is natural to dualize the Hurewicz map

$$h : \pi_n(X) \longrightarrow H_n(X)$$

and concentrate on the dual Hurewicz map

$$h^\vee : H^*(X) = H_*(X)^\vee \to \pi_*(X)^\vee.$$

In fact, the dual Hurewicz map $h^\vee$ must be zero on decomposable elements of H^*X, and so it yields a map from the indecomposable quotient of $H^*(X)$ to $\pi_*(X)^\vee$.

If $(\Lambda V, d)$ is a minimal Sullivan model for X then this dual Hurewicz map is the linear map

$$h^\vee : H^*(\Lambda V, d) \to V$$

that comes from dividing $(\Lambda V, d)$ by the sub-cochain complex $(\Lambda^{\geq 2} V, d)$. An element $x \in V$ is in the image of $h^\vee$ if and only if there is a $g \in \Lambda^{\geq 2} V$ such that $d(x + g) = 0$.

12.C. **The dual Hurewicz map and the Noetherian condition.** It is immediate from Theorem 8.1 that if X is sci then $h^\vee$ is an epimorphism in even degrees. The critical step in showing that gci implies sci is to prove a special case of this surjectivity holds for gci spaces. We are grateful to S.Iyengar for pointing out that a corresponding result with a very different proof appears as a crucial lemma in [11].

Proposition 12.2. *Suppose X is a rational space with finite dimensional homotopy and that the top degree in which homotopy is nonzero is $2n$. If $H^*(X)$ is Noetherian then the dual Hurewicz map*

$$h^\vee : H^{2n}(X) \longrightarrow \pi_{2n}^\vee(X)$$

is surjective.

Proof: By killing homotopy groups, there is a fibration sequence

$$Y \leftarrow X \stackrel{\Phi}{\leftarrow} K(\mathbb{Q}, 2n)$$

so that Φ a monomorphism in homotopy. We suppose $S = (\Lambda W, d)$ is a minimal Sullivan model for Y, and $R = (\Lambda V, d)$ is a minimal Sullivan model for X, where $V = W \oplus \mathbb{Q}x$. We can take $Q = (\Lambda x, 0)$ as a Sullivan model for $K(\mathbb{Q}, 2n)$ with $\Phi : R \to Q$ being the obvious map, so that

$$S \to R \stackrel{\Phi}{\to} Q \simeq R \otimes_S \mathbb{Q}.$$

provides an algebraic model of the fibration.

To conclude, we apply the following lemma.

Lemma 12.3. *If $H^*(\Phi)$ is non-trivial then x is in the image of the dual Hurewicz map.*

Proof: Suppose

$$x^n \in \mathrm{im}(H^*(X) \to H^*(K(\mathbb{Q}, n)) = \mathbb{Q}[x]).$$

This implies there is a cocycle in R of the form:

$$x^n + f_1 x^{n-1} + f_2 x^{n-2} + \cdots + f_n$$

with $f_i \in \Lambda W$. The differential of this cocycle is

$$(ndx + df_1)x^{n-1} + \text{ [terms of lower degree in } x \text{]} = 0,$$

which implies $ndx + df_1 = 0$. Hence there is an element $g \in R$ such that $x + g$ is a cocycle in R. The element $x + g$ cannot be a coboundary because R is minimal, and hence x is in the image of the dual Hurewicz map. □

If $H^*(X)$ is Noetherian, then by [13, 9.3], the stable Koszul complex can be used to construct the $\mathbb{Q}$-cellularization and there is a local cohomology spectral sequence

$$H_I^{-p}(H^*(Q))_q \;\Rightarrow\; H_{p+q}(\mathrm{Cell}_{\mathbb{Q}}^R(Q)),$$

where Q is considered as an R-module via Φ.

We now suppose x is not in the image of the dual Hurewicz map and deduce two contradictory statements about $\mathrm{Cell}_{\mathbb{Q}}^Q(Q)$. First, Lemma 12.3 implies that $H^*(\Phi)$ is trivial and hence the spectral sequence collapses at the E^2-page to show

$$\mathrm{Cell}_{\mathbb{Q}}^R(Q) \simeq Q.$$

In particular $\mathrm{Cell}_{\mathbb{Q}}^R(Q)$ has cohomology only in codegrees ≥ 0.

On the other hand, if we assume x is not in the image of the dual Hurewicz map, we will see that the cohomology of $\mathrm{Cell}_{\mathbb{Q}}^R(Q)$ must be quite different.

Lemma 12.4. *If $K(\mathbb{Q}, 2n) \longrightarrow X \longrightarrow Y$ is a fibration sequence killing the top homotopy group of X then $C^*(\Omega Y)$ is $\mathbb{Q}$-cellular as a module over $C^*(K(\mathbb{Q}, 2n))$.*

Proof: Since the homotopy groups of ΩY are concentrated in degrees less than $2n-1$ and ΩY is a product of Eilenberg-MacLane spaces, the connecting map $\Omega Y \longrightarrow K(\mathbb{Q}, 2n)$ is null. It follows that the induced map in cohomology $\mathbb{Q}[x] = H^*(K(\mathbb{Q}, 2n)) \longrightarrow H^*(\Omega Y)$ factors through $\mathbb{Q}$, and hence that $C^*(\Omega Y)$ is $\mathbb{Q}$-cellular as required. □

The third author has identified two elementary but very useful base change results for cellularization.

Lemma 12.5. (Independence of base [28, 3.1]**)** *Suppose $R \longrightarrow S$ is a map of rings.*
(i) **(Strong)** *If B is an S-module and $S \otimes_R B$ is B-cellular over S, then for any S-module Y*

$$\mathrm{Cell}_B^R Y \simeq \mathrm{Cell}_B^S Y.$$

*(ii)***(Weak)** *If A is an R-module and $S \otimes_R A$ is A-cellular over R, then for any S-module Y*

$$\mathrm{Cell}_A^R Y \simeq \mathrm{Cell}_{S\otimes_R A}^S Y. \quad \square$$

It follows from Lemma 12.4 that $Q \otimes_R \mathbb{Q}$ is $\mathbb{Q}$-cellular as an Q-module. We may therefore apply the Strong Independence of Base property to the map $R \to Q$ with $B = \mathbb{Q}$ and conclude

$$\mathrm{Cell}_{\mathbb{Q}}^R(Q) \simeq \mathrm{Cell}_{\mathbb{Q}}^Q(Q).$$

Since $H^*(Q) = \mathbb{Q}[x]$, we can easily compute

$$H^*(\mathrm{Cell}_{\mathbb{Q}}^Q(Q)) = \Sigma H_I^1(\mathbb{Q}[x]) = \Sigma \mathbb{Q}[x]^\vee$$

using the spectral sequence above. In particular $\mathrm{Cell}_{\mathbb{Q}}^Q Q$ has cohomology in negative codegrees. It is therefore not equivalent to Q, and the assumption that x is not in the image of the dual of the Hurewicz map leads to a contradiction.

This completes the proof of the proposition. □

This completes the proof of Theorem 12.1. □

13. Examples.

It is quite easy to construct examples in rational homotopy theory, so we can see that various classes are distinct.

13.A. **Homotopy invariant notions and cohomology rings.** We may impose a homotopy invariant condition on a space X or a conventional condition on the cohomology ring $H^*(X)$.

In the regular case there is no distinction by Theorem 5.4, since rational graded connected commutative rings are regular if and only if they are polynomial on even degree generators.

In the ci case the homotopy invariant notion is strictly weaker than the notion for cohomology rings. Indeed, we show in Proposition 13.1 that if $H^*(X)$ is ci then X is sci. On the other hand, Example A.6 gives a pure sci space whose cohomology ring is not even Gorenstein.

Proposition 13.1. *If $H^*(X)$ is a complete intersection, then X is formal, and there is a fibration*

$$S^{m_1} \times \cdots \times S^{m_c} \longrightarrow X \longrightarrow KV$$

with $m_1, m_2, \dots, m_c$ odd. In particular, X is also sci.

Remark 13.2. By Theorem 8.1 a general sci space X has a similar fibration with fibre an arbitrary space with finite dimensional odd homotopy. Those with $H^*(X)$ ci also have zero Postnikov invariants: there are vastly more sci spaces than those with ci cohomology. However Example A.6 shows that even when the fibre is a product of odd spheres, the cohomology ring need not be ci.

Proof: We may suppose $H^*(X) = k[x_1, \dots, x_n]/(f_1, \dots, f_c)$ for suitable even degree generators $x_1, \dots, x_n$ and regular sequence $f_1, \dots, f_c$. Now let V be a graded vector space with basis $x_1', \dots, x_n'$, where the degree of x_i' is the same as that of x_i, and let W be a graded vector space with basis $\phi_1', \dots, \phi_c'$ where the codegree of ϕ_i' is one less than that of f_i. We now take X' to have model $M(X') = (\Lambda(V) \rtimes \Lambda(W), d\phi_1' = f_1', \dots, d\phi_c' = f_c')$. Since $f_1, \dots, f_c$ is a regular sequence, $H^*(X') \cong H^*(X)$.

Now construct a map

$$g : M(X') \longrightarrow C^*(X)$$

by taking $g(x_i')$ to be a representative cycle for $x_i \in H^*(X)$. Since f_i is trivial in $H^*(X)$, we may choose $\phi_i \in C^*(X)$ so that $d\phi_i = f_i$, and define $g(\phi_i') = \phi_i$. The resulting map g is a cohomology isomorphism and therefore an equivalence.

The structure of $M(X')$ gives a fibration as claimed. $\square$

Similarly, in the Gorenstein case the homotopy invariant notion is strictly weaker. On the one hand, Corollary A.3 shows that if $H^*(X)$ is Gorenstein, then X is h-Gorenstein. If X is h-Gorenstein and $H^*(X)$ is Cohen-Macaulay then $H^*(X)$ is Gorenstein. However Example A.6 gives an h-Gorenstein space whose cohomology ring is not Cohen-Macaulay.

13.B. **Separating the hierarchy.** Since h-regular spaces are of the form KV, it is easy to see that there are sci spaces that are not regular. To give an example of an h-Gorenstein space that is not gci, we may use connected sums of manifolds, as in Example 13.3. Finally, there are many spaces with Noetherian cohomology that are not h-Gorenstein: two easy sources of examples are either finite dimensional spaces whose cohomology ring does not satisfy Poincaré duality, or Cohen-Macaulay rings which are not Gorenstein.

Example 13.3. We provide a space X with $H^*(X)$ so that X is h-Gorenstein but not gci. Almost any non-trivial connected sum of manifolds will do, but we give an explicit example.

First, note that if M and N are manifolds, their connected sum

$$M\#N = (M' \vee N') \cup e^n$$

where M' is M with a small disc removed, and similarly for N'. By considering Lie models as in [18, 24.7], we obtain

$$\pi_*(\Omega(M\#N)) = (\pi_*(\Omega M') * \pi_*(\Omega N'))/(\alpha + \beta),$$

where $*$ is the coproduct of graded Lie algebras, and where α and β are the attaching maps for the top cells in M and N.

Perhaps the simplest thing to try is $M = N = \mathbb{C}P^2$. Here we obtain

$$\pi_*(\Omega(\mathbb{C}P^2\#\mathbb{C}P^2)) = \mathrm{Lie}(u_1, v_1)/([u_1, u_1] + [v_1, v_1]),$$

where $\mathrm{Lie}(V)$ denotes the free graded Lie algebra on V. This shows $\pi_*(\Omega(\mathbb{C}P^2\#\mathbb{C}P^2))$ is finite, so $\mathbb{C}P^2\#\mathbb{C}P^2$ is gci, which also follows from the fact that its cohomology ring

$$H^*(\mathbb{C}P^2\#\mathbb{C}P^2) = \mathbb{Q}[a_2, b_2]/(a^2, b^2)$$

is a complete intersection.

However, once we take three copies, we obtain

$$\pi_*(\Omega(\mathbb{C}P^2\#\mathbb{C}P^2\#\mathbb{C}P^2)) = \mathrm{Lie}(u_1, v_1, w_1)/([u_1, u_1] + [v_1, v_1] + [w_1, w_1]),$$

which is not finite, so that $\mathbb{C}P^2\#\mathbb{C}P^2\#\mathbb{C}P^2$ is not gci, giving the required example.

Example 13.4. Here is an example of a space which is gci but not zci. Let X be the space with model $R = (\Lambda(u_3, v_3, w_5), dw = uv)$. There is a map of graded rings $\Psi : Z\mathbf{D}(R) \to H_*(\Omega X)$ given by

$$\Psi(\zeta) \ = \ (\zeta_{\mathbb{Q}} : \mathbb{Q} \to \Sigma_n\mathbb{Q}) \in H_n(\Omega X)$$

Clearly the image of Ψ is contained in the center of $H_*(\Omega X)$. The graded ring $H_*(\Omega X)$ is the enveloping algebra of the graded Lie algebra L, where L is generated by three elements U_2, V_2 and W_4 and a single relation $UV = W$. The center of $H_*(\Omega X)$ is therefore the set $\{\mathbb{Q}W^n | n \geq 0\}$. Now suppose X was zci, then we would have had appropriate elements $\zeta_1, ..., \zeta_n \in Z\mathbf{D}(R)$. Since the degree of ζ_i is non zero, $\Psi(\zeta_i)$ is either zero or $a_i W^{n_i}$ for some $n_i > 0$.

Let M be the R-module that is the cone of the map $\mathbb{Q} \xrightarrow{W} \Sigma_4\mathbb{Q}$. The module $M/\zeta_1/\cdots/\zeta_n$ must be a small R-module. Now consider the $C_*(\Omega X)$-module $\bar{M} = \mathrm{Hom}_R(\mathbb{Q}, M)$. By the Yoneda lemma, the map

$$\bar{\zeta}_i = \mathrm{Ext}^*_R(\mathbb{Q}, \zeta_i) : H_*(\bar{M}) \to H_{*+|\zeta_i|}(\bar{M})$$

of $H_*(\Omega X)$-modules is simply multiplication by $\Psi(\zeta_i)$. It is easy to see that

$$H_*(\bar{M}) = H_*(\Omega X)/(W) = \mathbb{Q}[U, V]$$

and therefore the induced map $\bar{\zeta}_i$ is zero on the homology of $\bar{M}$. We conclude that $\bar{M}/\bar{\zeta}_1/\cdots/\bar{\zeta}_n$ has infinitely many nonzero homology groups. However,

$$\bar{M}/\bar{\zeta}_1/\cdots/\bar{\zeta}_n \simeq \mathrm{Hom}_R(Q, M/\zeta_1/\cdots/\zeta_n)$$

and since $M/\zeta_1/\cdots .../\zeta_n$ is small and $C^*(X)$ is h-Gorenstein we see that the $C_*(\Omega X)$-module $\mathrm{Hom}_R(Q, M/\zeta_1/\cdots/\zeta_n)$ has only finitely many nonzero homology groups, in contradiction.

13.C. **Miscellaneous examples.** The following example shows that the Noetherian condition is essential in the definition of gci.

Example 13.5. The space X with model $(\Lambda(v_{2a}, x_{2b+1}, w_{2a+2b}), dw = vx)$ is not sci. For definiteness, we work with $a = b = 1$, so that we have the model $(\Lambda(v_2, x_3, w_4), dw = vx)$.

Indeed, if X is sci it must be in a fibration $S^3 \longrightarrow X \longrightarrow KV$ where $V = \mathbb{Q}\{v, w\}$. But then in homotopy we have a short exact sequence

$$0 \longrightarrow \pi_*(\Omega S^3) \longrightarrow \pi_*(\Omega X) \longrightarrow \pi_*(\Omega KV) \longrightarrow 0$$

of graded Lie algebras, which implies that $\pi_*(\Omega S^3)$ is an ideal of $\pi_*(\Omega X)$. On the other hand, since $dw = vx$, the corresponding elements $\overline{v}_1, \overline{x}_2$ and $\overline{w}_3$ in the Lie algebra $\pi_*(\Omega X)$ satisfy $\overline{w} = [\overline{v}, \overline{x}]$, and we have a contradiction since $\pi_*(\Omega S^3)$ is generated by $\overline{x}$.

This is consistent with our general results since the cohomology ring is not Noetherian. Indeed, the cohomology ring $H^*(X)$ is $\mathbb{Q}[v]$ in even degrees, whilst all products of the odd degree elements $x, wx, w^2x, \ldots$ are zero.

14. The nci condition

In this section we conclude by giving a condition in the style of the zci condition which captures polynomial growth in the non-Noetherian situation. From another point of view, since Example 13.5 shows that the Noetherian condition is essential, the nci condition introduced here is genuinely weaker than both the zci and the eci conditions. The letter 'n' in nci stands for nilpotent.

14.A. **The condition.** We suppose that R is a CDGA and continue to write $\mathbf{D}(R)$ for the derived category of dg-R-modules.

Definition 14.1. We say that R is *nci of length* $\leq n$ if there are

(1) a sequence of triangulated subcategories $\mathbf{D}(R) = \mathbf{D}_0 \supseteq \mathbf{D}_1 \supseteq \cdots \supseteq \mathbf{D}_n$ (not necessarily full) and
(2) natural transformations $\zeta_i : 1_{\mathbf{D}_i} \to \Sigma_{|\zeta_i|} 1_{\mathbf{D}_i}$ for $i = 0, ..., n-1$

such that the following conditions hold

(1) Every $\mathbf{D}_i$ is closed under coproducts.
(2) Every ζ_i is central among natural transformations of $1_{\mathbf{D}_i}$.
(3) For every $X \in \mathbf{D}_i$ there exists an object $X/\zeta_i \in \mathbf{D}_{i+1}$ and a distinguished triangle $X \xrightarrow{\zeta_i} \Sigma_{|\zeta_i|} X \to X/\zeta_i$.
(4) If $0 \neq X \in \mathbf{D}_i$ is in the thick subcategory generated by $\mathbb{Q}$ then X/ζ_i is non-zero.
(5) If $0 \neq X \in \mathbf{D}(R)$ is in the thick subcategory generated by $\mathbb{Q}$ then $X/\zeta_0/\zeta_1/\cdots/\zeta_{n-1}$ is non-zero and small as an object of $\mathbf{D}(R)$.

Remark 14.2. (i) Note first that if R is zci or eci of codimension c, it is clearly nci of length c.

(ii) On the other hand, if R is nci and the natural transformations $\zeta_1, ..., \zeta_{n-1}$ can be extended to central natural transformations of $1_{\mathbf{D}(R)}$, then R is almost zci (the only additional condition required is that the cohomology be Noetherian). Remark 14.11 below provides an explicit example where it is not possible to extend one of these natural transformations.

In the context of rational homotopy theory there is a straightforward characterization of nci CDGAs.

Theorem 14.3. *Let $R = (\Lambda V, d)$ be a minimal Sullivan algebra. Then R is nci if and only if V is finite dimensional.*

Lemma 14.5 below shows that if V is infinite dimensional, then R is not nci. The converse is proved in Subsection 14.G below.

Remark 14.4. The contrast with eci spaces, where Theorem 8.1 shows the structure is much more constrained (the differential on even generators is zero) is very striking.

On the other hand, the only difference is the Noetherian condition. Indeed, if X is nci and $H^*(X)$ is Noetherian, then Theorem 14.3 shows that X is gci and therefore (by Theorems 12.1 and 11.1) also eci.

14.B. **Exponential Growth.** The hard work in this section is in dealing with the case of a natural transformation of degree zero. This may be a useful counterpart to the approach to the Jacobson radical in [9].

Lemma 14.5. *Let X be a simply-connected finite CW-complex. If $C^*(X)$ is nci, then $H_*(\Omega X)$ has polynomial growth.*

Proof: Since $R = C^*(X)$ is nci there is a sequence of R-modules $M_0, M_1, ..., M_n$ such that

(1) $M_0 = \mathbb{Q}$ and $M_i \in \mathbf{D}_i$.
(2) There is a distinguished triangle $\Sigma^{|\zeta_i|} M_i \xrightarrow{\zeta_i} M_i \to M_{i+1}$.
(3) M_n is a small R-module.

Suppose, by way of contradiction, that $H^*(\Omega X)$ has exponential growth. As in Section 9, if all the degrees $|\zeta_0|, ..., |\zeta_{n-1}|$ are non-zero, then we have a contradiction, since by Lemma 9.3 $H^*(M_n \otimes_R \mathbb{Q})$ must have exponential growth because $H^*(M_0 \otimes_R \mathbb{Q}) \cong H^*(\Omega X)$ has exponential growth.

We are left with showing that if $|\zeta_i| = 0$ for some i, the exponential growth still propagates. So suppose that $|\zeta_i| = 0$ for some i and $M_i \otimes_R \mathbb{Q}$ has exponential growth. Consider the homotopy colimit M_i^∞ of the telescope $M_i \xrightarrow{\zeta_i} M_i \xrightarrow{\zeta_i} \cdots$. Since this homotopy colimit is part of a distinguished triangle: $\oplus_{n=0}^\infty M_i \xrightarrow{1-\zeta_i} \oplus_{n=0}^\infty M_i \to M_i^\infty$ where the first map is in $\mathbf{D}_i$, M_i^∞ is also in $\mathbf{D}_i$.

Next we show that M_i^∞ is finitely built from $\mathbb{Q}$. By construction $H^*(M_i)$ is non-zero only in finitely many degrees. Each cohomology group $H^j(M_i)$ is a finite dimensional vector space on which ζ_i acts as a linear transformation. For large enough m, the kernel of ζ_i^m stabilizes. Denote this kernel by $K \subseteq H^j(M_i)$, so that we can write

$$H^j(M_i) \cong K \oplus V,$$

where ζ_i^m is zero on K and is an isomorphism on V. We see that $H^j(M_i^\infty) \cong V$ is also finite dimensional, and M_i^∞ itself is finitely built from $\mathbb{Q}$.

The morphism $M_i^\infty \xrightarrow{\zeta_i} M_i^\infty$ is an equivalence, so Condition (4) of the definition of nci shows $M_i^\infty \simeq 0$. As $M_i^\infty \otimes_R \mathbb{Q} \simeq 0$ and $H^j(M_i \otimes_R \mathbb{Q})$ is finite dimensional for each j, it follows that $H^j(\zeta_i \otimes_R \mathbb{Q})$ is nilpotent on $H^j(M_i \otimes_R \mathbb{Q})$ for each j, i.e., for each j there exists an n such that $\zeta_i^n \otimes_R \mathbb{Q}$ induces the zero map on $H^j(M_i \otimes_R \mathbb{Q})$.

First observe that $[M_i, M_i]_R$ (i.e., the ring of degree 0 homotopy endomorphisms) is a finite dimensional algebra. Indeed, we start by observing that $[M_0, M_0]_R^* = [\mathbb{Q}, \mathbb{Q}]_R^*$ is finite dimensional in each degree, and deduce the same for $[M_1, M_1]_R^*, [M_2, M_2]_R^*, \ldots, [M_i, M_i]_R^*$ using the defining triangles.

Now let z be the morphism $M_i \otimes_R \mathbb{Q} \xrightarrow{\zeta_i \otimes_R \mathbb{Q}} M_i \otimes_R \mathbb{Q}$, and note that the span of $\{z, z^2, z^3 ...\}$ inside $[M_i \otimes_R \mathbb{Q}, M_i \otimes_R \mathbb{Q}]_{C_*(\Omega X)} \cong [M_i, M_i]_R$ is finite dimensional. We now resort to a classical trick to show that z is nilpotent on $H^*(M_i \otimes_R \mathbb{Q})$, i.e., that there is some N for which z^N induces the zero map on $H^*(M_i \otimes_R \mathbb{Q})$.

Suppose $\{z, z^2, ..., z^n\}$ is a basis for the span of $\{z, z^2, z^3, ...\}$. Let $x \in H^j(M_i \otimes_R \mathbb{Q})$ for some j and let $\{z(x), z^2(x), ..., z^k(x)\}$ be a basis for the span of $\{z(x), z^2(x), z^3(x), ...\}$.

Clearly $k \leq n$. Suppose that $z^m(x) \neq 0$ and $z^{m+1}(x) = 0$, so that $m \geq k$. We will show that $m = k$ so $z^{k+1}(x) = 0$.

Write

$$z^m(x) = a_1 z(x) + a_2 z^2(x) + \cdots + a_j z^j(x),$$

where $a_j \neq 0$ and $j \leq k$. First, we note that $j = k$. Indeed, applying z we find $0 = z^{m+1}(x) = \sum_{i=1}^{j} a_i z^{i+1}(x)$, so that if $j < k$ we get a linear dependence. Thus, for some $t \leq k$ we have

$$z^m(x) = a_t z^t(x) + a_{t+1} z^{t+1}(x) + \cdots + a_k z^k(x),$$

where $a_t \neq 0$. It suffices to show $t = k$, so we suppose $t < k$ and deduce a contradiction. If $t < k$ we apply z to our equation and deduce

$$z^{m+1}(x) = 0 = a_t z^{t+1}(x) + a_{t+1} z^{t+2}(x) + \cdots + a_{n-1} z^k(x) + a_k z^{k+1}(x)$$

which means that $z^{k+1}(x)$ is in the span of $\{z^{t+1}(x), ..., z^k(x)\}$. Applying z repeatedly, we deduce z^{k+s} is also in the span of $\{z^{t+1}(x), ..., z^k(x)\}$ for all $s \geq 1$. Now either $m = k$ and we are done, or $m = k + s$ for some $s \geq 1$ and we obtain a contradiction. Accordingly, $t = k$ and $z^{k+1}(x) = 0$ as required.

Since $H^*(M_i \otimes_R k)$ has exponential growth, it follows that the kernel of the map $H^*(z) : H^*(M_i \otimes_R k) \to H^*(M_i \otimes_R k)$ has exponential growth. In particular, this implies that the cone of $z = \zeta_i \otimes \mathbb{Q} : M_i \otimes_R \mathbb{Q} \to M_i \otimes_R \mathbb{Q}$ has exponential growth. □

14.C. **The first unravelling move.** We now describe three constructions we may use to build a new nci space X from a given nci space X'. In practice we are given X, and we unravel the process to obtain X' in such a way that if X' is nci, so too is X. Only the last of these three was necessary in the eci case (it was the critical role of Proposition 12.2 to show this). We work entirely algebraically, so that R is a model for X and R' is a model for X'.

The first move is eliminating an even generator that is also a cocycle.

Lemma 14.6. *Let $R = (\Lambda V, d)$ be a minimal Sullivan algebra and let $x \in V$ be an element of even degree such that $dx = 0$. Then multiplication by x yields a natural transformation on $\mathbf{D}(R)$: $M \xrightarrow{x\cdot} \Sigma^{-|x|} M$ whose cone is $R/(x) \otimes_R M$. If $R/(x)$ is nci of length $\leq n$ then R is nci of length $\leq n+1$.*

We record the topological counterpart of this lemma.

Lemma 14.7. *Let $x \in \pi_{2n}^{\vee}(X)$ be an element that is in the image of the dual Hurewicz map $H^*(X) \to \pi_*^{\vee}(X)$. Then there is a fibration sequence:*

$$X' \to X \to K(\mathbb{Q}x)$$

such that x is in the image of $\pi_{2n}^{\vee}(K(\mathbb{Q}x))$. If X' is nci of length $\leq n$ then X is is nci of length $\leq n+1$.

Proof of Lemma 14.6: In this case there is a short exact sequence of dg-R-modules:

$$\Sigma^{|x|} R \hookrightarrow R \twoheadrightarrow R/(x).$$

The leftmost map is given by multiplication by x: $a \mapsto a \cdot x$. We write $V = W \oplus \mathbb{Q}x$ so that $R/(x) \cong (\Lambda W, d)$, isomorphic to a sub-DGA of R.

Define a natural transformation $\zeta : 1_{\mathbf{D}(R)} \to \Sigma^{-|x|} 1_{\mathbf{D}(R)}$ as multiplication by x. If M is a cofibrant dg-R-module, then applying $- \otimes_R M$ to the short exact sequence above yields a distinguished triangle

$$\Sigma^{|x|} M \xrightarrow{\zeta} M \to R/(x) \otimes_R M$$

in $\mathbf{D}(R)$.

Now suppose that $R/(x)$ is nci of length $\leq n$, so that there are subcategories $\mathbf{D}_{R/(x)} = \mathbf{D}_0 \supseteq \mathbf{D}_1 \supseteq \cdots \supseteq \mathbf{D}_n$ and appropriate natural transformations $\zeta_0, ..., \zeta_{n-1}$. The map $p : R \to R/(x)$ of CDGAs induces an obvious functor $p^* : \mathbf{D}(R/(x)) \to \mathbf{D}(R)$. Define $\mathbf{D}'_i = p^* \mathbf{D}_i$ and similarly $\zeta'_i = p^* \zeta_i$. Set $\mathbf{D}'_{-1} = \mathbf{D}(R)$ and let ζ'_{-1} be the natural transformation ζ defined above.

We may check that the subcategories $\mathbf{D}'_{-1}, ..., \mathbf{D}'_{n-1}$ and natural transformations $\zeta'_{-1}, ..., \zeta'_{n-1}$ satisfy the conditions for being nci. One need only note three things. First, if M is a small dg-$R/(x)$-module then $p^* M$ is a small dg-R-module, because $R/(x)$ is a small R-module. Second, if N is a dg-R-module finitely built by $\mathbb{Q}$, then so is $R/(x) \otimes_R N$. The reason is that N is finitely built by $\mathbb{Q}$ over R if and only if $H^*(N)$ is finite dimensional. Third, if $N \neq 0$ is finitely built by $\mathbb{Q}$, then N/ζ is not zero, because $|\zeta| \neq 0$ and so ζ cannot induce an isomorphism of $H^*(N)$. $\square$

14.D. **The second unravelling move.** The second move eliminates an even cocycle by adding an odd generator. The argument is essentially the same as the previous one (except that the cocycle is not a generator), so we omit the proof.

Lemma 14.8. *Let $R = (\Lambda V, d)$ be a minimal Sullivan algebra and let $f \in R$ be an even cocycle. Let $R' = (\Lambda(V \oplus \mathbb{Q}y), d')$, where $d'y = f$ and $d'v = dv$ for all $v \in V$. Then multiplication by f yields a natural transformation on $\mathbf{D}(R)$: $M \xrightarrow{f\cdot} \Sigma^{-|f|} M$ whose cone is $R' \otimes_R M$. If R' is nci of length $\leq n$ then R is is nci of length $\leq n+1$.*

The topological counterpart of this lemma is again a fibration:

$$X' \to X \to K(\mathbb{Q}f),$$

only this time we just require that $X \to K(\mathbb{Q}f)$ represent a nontrivial element in $H^*(X)$.

14.E. **The third unravelling move.** Finally, the third move is passing to a subalgebra. This is the precise counterpart of the argument of Subsection 12.A, and the conclusion is analogous to a spherical fibration

$$S^{|x|-1} \longrightarrow X \longrightarrow X'.$$

Nevertheless, we describe how this move fits within the nci context.

Lemma 14.9. *Let $R = (\Lambda V, d)$ be a minimal Sullivan algebra. Let $x \in V$ be an element of odd degree such that $dv \notin x\Lambda V$ for all $v \in V$. Then there is a sub Sullivan algebra $Q \subset R$ and a natural transformation $\zeta : 1_{\mathbf{D}(R)} \to \Sigma^{|x|+1} 1_{\mathbf{D}(R)}$ such that for any $M \in \mathbf{D}(R)$ there is a distinguished triangle in $\mathbf{D}(R)$:*

$$M \xrightarrow{\zeta} \Sigma^{|x|+1} M \to \Sigma R \otimes_Q M\ .$$

If Q is nci of length $\leq n$ then R is nci of length $\leq n+1$.

Proof: Write $V = W \oplus \mathbb{Q}x$, and let Q be the Sullivan algebra $(\Lambda W, d)$. Clearly Q is a minimal Sullivan algebra and there is an obvious inclusion $\iota : Q \hookrightarrow R$ of Sullivan algebras. We have seen earlier that there is a natural transformation ζ on $\mathbf{D}(R)$ such that $M \xrightarrow{\zeta} \Sigma^{|x|+1}M \to \Sigma R \otimes_Q M$ is a distinguished triangle for all $M \in \mathbf{D}(R)$.

Suppose that Q is nci of length $\leq n$, so there are subcategories $\mathbf{D}_Q = \mathbf{D}_0 \supseteq \mathbf{D}_1 \supseteq \cdots \supseteq \mathbf{D}_n$ and appropriate natural transformations $\zeta_0, ..., \zeta_{n-1}$. The map $\iota : Q \to R$ of CDGAs induces a functor $\iota_* : \mathbf{D}(Q) \to \mathbf{D}(R)$, where $\iota_*(N)$ is the induced dg-R-module $R \otimes_Q N$. Define $\mathbf{D}'_i = \iota_* \mathbf{D}_i$ and similarly $\zeta'_i = \iota_* \zeta_i$. Set $\mathbf{D}'_{-1} = \mathbf{D}(R)$ and let ζ'_{-1} be the natural transformation ζ on $\mathbf{D}(R)$ defined above.

We may check that the subcategories $\mathbf{D}'_{-1}, ..., \mathbf{D}'_{n-1}$ and natural transformations $\zeta'_{-1}, ..., \zeta'_{n-1}$ satisfy the conditions for being nci. One need only note three things. First, if N is a small DG-Q-module then $\iota_* N$ is a small DG-R-module. Second, if M is a dg-R-module finitely built by $\mathbb{Q}$, then M is finitely built by $\mathbb{Q}$ also as a dg-Q-module. Third, $|\zeta| \neq 0$ and therefore $M/\zeta \neq 0$ for every non-zero dg-R-module M that is finitely built by $\mathbb{Q}$. $\square$

14.F. **Two examples.** We discuss some examples of minimal Sullivan algebras which are nci but not eci.

Example 14.10. Consider the minimal Sullivan algebra

$$R = (\Lambda(x_3, y_3, z_3, a_8), dx = dy = dz = 0, da = xyz).$$

First, we see that it is nci by showing explicitly how to unravel it. Indeed, we may apply Lemma 14.8 to the cocycle xy to yield

$$R' = (\Lambda(x, y, z, w, a), da = xyz, dw = xy).$$

Now, $d(wz) = da$, so by a change of variables $a' = a - wz$ we see that

$$R' \cong (\Lambda(x, y, z, w, a'), da' = 0, dw = xy)$$

Now R' is eci and from its homotopy we see it is of codimension 4. It is therefore also nci of length 4, and therefore R is nci of length ≤ 5.

On the other hand, it is not hard to see that R is not eci. Most explicitly, one may identify the cohomology ring explicitly and observe that it is not Noetherian: it has a basis of monomials $x^i y^j z^k a^l$ where $i, j, k \in \{0, 1\}$ omitting the monomials $xyza^l$ for $l \geq 0$ and a^l for $l \geq 1$. All elements (except those in codegree zero) are nilpotent.

Note also that the dual Hurewicz map is not surjective in codegree 8, so that the method of Subsection 12.A cannot be applied.

Remark 14.11. Finally, we can see explicitly why the natural transformation in the nci definition cannot always be extended as we would require for the zci definition. In the previous example, multiplication by the cocycle $a' = a - wz$ defines a natural transformation on the $\mathbf{D}(R')$ and therefore also on the image of $\mathbf{D}(R')$ under restriction. This natural transformation cannot be extended to a central natural transformation of $1_{\mathbf{D}(R)}$, since we

would then have a commutative diagram

$$\begin{tikzcd}
\Sigma^{|xy|+|a'|}R \arrow[r, "xy"] \arrow[d, "a'"] & \Sigma^{|a'|}R \arrow[r] \arrow[d, "a'"] & \Sigma^{|a'|}R' \arrow[d, "a'"] \\
\Sigma^{|xy|}R \arrow[r, "xy"] \arrow[d] & R \arrow[r] \arrow[d] & R \arrow[d, "a'"] \\
\Sigma^{|xy|}B \arrow[r, "xy"] & B \arrow[r] & B'
\end{tikzcd}$$

of distinguished triangles in $\mathbf{D}(R)$. The natural transformation a' must be the zero map on R, hence $B \cong R \oplus \Sigma^{|a|}R$. This shows that B' is isomorphic to $R' \oplus \Sigma^{|a|}R'$, and therefore a' acts as a regular element. However this contradicts the fact that $(a')^2 = 0$ from the long exact sequence of the triangle.

Example 14.12. Consider the minimal Sullivan algebra

$$R = (\Lambda(x_5, y_3, z_3, y_3', z_3', a_{10}), dx = yz + y'z', da = xyy').$$

First, we see that it is nci by showing explicitly how to unravel it. We apply Lemma 14.8 to the cocycle yy', yielding:

$$R' = (\Lambda(x_5, y_3, z_3, y_3', z_3', a_{10}, w_5), dx = yz + y'z', da = xyy', dw = yy')$$

This yields $d(a + xw) = (dx)w$. We apply Lemma 14.8 twice more, for the cocycles wy and wy', yielding:

$$R'' = (\Lambda(x_5, y_3, z_3, y_3', z_3', a_{10}, w_5, t_7, t_7'), dx = yz+y'z', da = xyy', dw = yy', dt = wy, dt' = wy')$$

Finally we have $d(a + xw - zt - zt') = 0$. So, as in the previous example, we can do a change of variables $a' = a + xw - zt - zt'$ and see that R'' is eci of codimension 8, and hence nci of length 8. It follows that R is *nci* of length ≤ 11.

On the other hand, it is not hard to see that R is not eci. Indeed, since the differential is non-zero on the top even class a_{10}, the dual Hurewicz map is not surjective in codegree 10, so that the method of Subsection 12.A cannot be applied. By Proposition 12.2, $H^*(R)$ is not Noetherian and so R is not eci.

14.G. **Proof of Theorem 14.3.** Let $R = (\Lambda V, d)$ be a simply-connected minimal Sullivan algebra where V is finite dimensional. We will show that R is nci. The proof proceeds by induction on the dimension of V^{even}. The induction starts since if $V^{\text{even}} = 0$, successive applications of the third unravelling move (Lemma 14.9) will reduce to a Sullivan algebra with trivial differential. This is then the model of a product of odd spheres, which is obviously nci.

If $V^{\text{even}} \neq 0$, then we apply Lemma 14.14 below, which says we may repeatedly apply the second unravelling move (i.e., add a finite number of odd generators) until we reach a CDGA R' with an even generator $a \in V^{\text{even}}$ such that $da = 0$. Now use the first unravelling move (Lemma 14.6) on a. The minimal Sullivan algebra $R'/(a)$ is nci by the inductive hypothesis, so that R' is nci by Lemma 14.6, and R is nci by Lemma 14.8. $\square$

The key ingredient is the following technical result. Note that for a minimal Sullivan algebra $R = (\Lambda V, d)$ with V of finite type, the image of an element $[f] \in H^n(R)$ under the

dual Hurewicz $h^\vee$ map is non-zero if and only if there is an isomorphism of minimal Sullivan algebras $\rho : R \xrightarrow{\cong} (\Lambda V', d')$ such that $\rho(f) \in V'$.

Lemma 14.13. *Let $R = (\Lambda V, d)$ be a minimal (simply connected) Sullivan algebra such that V is finite dimensional and concentrated only in odd degrees. Let $0 \neq [f] \in H^{2i+1}(R)$ be an odd element of the cohomology of R. If $h^\vee([f]) = 0$, then there is an element $0 \neq [g] \in H^{2j}(R)$ such that $0 < j \leq i$.*

Proof: Suppose the image of $[f]$ under the dual Hurewicz map is zero. The proof goes by induction on the dimension of V. Let $x \in V$ be an element of minimal codegree, therefore $dx = 0$ and $h^\vee(x) \neq 0$. Eliminate x by adding an even generator $S = (\Lambda(V \oplus \mathbb{Q}a), da = x)$ (d being defined on V as before). There is a distinguished triangle in $\mathbf{D}(R)$:

$$\Sigma^{|x|} S \to R \xrightarrow{\varphi} S \xrightarrow{\psi} \Sigma^{|x|+1} S.$$

Since S is equivalent to the minimal Sullivan algebra $R/(x)$, the induction assumption holds for S. There are two possible cases.

In the first case $H_*\varphi[f] \neq 0$. The image of $\varphi[f]$ under the dual Hurewicz map is zero because $\pi_*^\vee R \to \pi_*^\vee S$ is an epimorphism with kernel $\mathbb{Q}x$. By the induction assumption there is a even degree element in the cohomology of S whose codegree is smaller than $|f|$. Let $[g]$ be such an element of minimal degree. If $[g]$ is in the image of φ, then we are done. If not, then $\psi[g] \neq 0$. But $\psi[g] \in H^{|g|-|x|+1}(S)$, which contradicts the minimality of $|g|$ (note that $|g| - |x| + 1 > 0$, since otherwise g is one degree below the minimal generators, which is impossible).

The remaining option is that $\varphi[f] = 0$. Hence $f = dw$ for some $w \in S$. Write w as

$$w = a^n A_n + a^{n-1} A_{n-1} + \cdots + a A_1 + A_0,$$

where $A_i \in \Lambda V$. Clearly f is homologous to $f - dA_0$, so without loss of generality we can assume that $A_0 = 0$. Also note that all the A_i are of even codegrees smaller than the codegree of f (in fact $|A_i| \leq |f| - |x|$). Calculating dw gives

$$dw = a^n d(A_n) + \sum_{i=1}^{n} a^{i-1}(dA_{i-1} + ixA_i).$$

Since $f \in \Lambda V$ we see that:

$$\begin{aligned} f &= xA_1 \\ dA_{i-1} &= -ixA_i \quad \text{for } i \geq 2 \\ dA_n &= 0 \end{aligned}$$

Thus A_n is a cocycle. If A_n is not a coboundary in R, then we are done. Otherwise there is a B_n so that

$$A_n = dB_n$$

Now $dA_{n-1} = -nxA_n = -nxdB_n = d(nxB_n)$, whence $A_{n-1} - nxB_n$ is an even codegree cocycle. Again, if it is not a coboundary then we are done. Otherwise there is a B_{n-1} so that

$$A_{n-1} - nxB_n = dB_{n-1}.$$

Now

$$dA_{n-2} = -(n-1)xA_{n-1} = -(n-1)x(nxB_n + dB_{n-1})$$
$$= -(n-1)xdB_{n-1} = (n-1)d(xB_{n-1}).$$

So we see that $A_{n-2} - (n-1)xB_{n-1}$ is an even codegree cocycle. We continue in this manner until either we get the desired even codegree element in the cohomology of R, or we end with

$$A_1 - 2xB_2 = dB_1.$$

But now $f = xA_1 = x(2xB_2 + dB_1) = xdB_1 = -d(xB_1)$, i.e. $[f] = 0$, which is a contradiction. □

Using the previous lemma we can now show that the second unravelling move may be used to make an even generator become a cycle.

Lemma 14.14. *Let $R = (\Lambda V, d)$ be a minimal (simply connected) Sullivan algebra such that V is finite dimensional. Then there is a sequence $R = R_0 \to R_1 \to \cdots \to R_n$ of unravelling moves of the second type (i.e., moves to which Lemma 14.8 applies) such that R_n is isomorphic to a minimal Sullivan algebra: $R_n' = (\Lambda V', \delta)$, where there is a minimal even element $a \in V'$ such that $\delta a = 0$.*

Proof: If there is a minimal even codegree element $a \in V$ such that $da = 0$ we are done. Otherwise, choose some minimal even codegree element $a \in V$. Let $V_s \subset V$ be the subspace of codegrees smaller than $||a||$. Minimality of the Sullivan algebra implies that $da \in \Lambda V_s$.

Apply the second unravelling move (Lemma 14.8) to the even elements of $H^*(R)$ starting from the bottom and going up. Hence, let $g \in R$ be a minimal even degree cocycle that is not a coboundary. Applying Lemma 14.8 to g yields

$$R_1 = (\Lambda(V \oplus \mathbb{Q}x_g), dx_g = g)$$

and a map $R \to R_1$. We continue adding odd generators to R in this way until there is no more homology in even codegrees less than or equal to $||a||$. Note that we need only kill cocycles composed of odd generators only, even in codegree a.

We end with a minimal Sullivan algebra $R_n = (\Lambda U, d)$, where R_n has no even degree cohomology in dimension $||a||$ or lower. Let $U_s \subset U$ be the subspace of codegrees smaller than $||a||$. Clearly U_s has only odd degree elements. Define $T = (\Lambda U_s, d)$ to be the appropriate sub-CDGA of R_n. Then T also has no cohomology in even codegrees $||a||$ or lower. Because da is a cocyle in T, $h^\vee(da) = 0$ and $H^{2j}(T) = 0$ for $2j \le ||a||$, it follows from Lemma 14.13 that da is a coboundary in T. Thus $da = du$ for some $u \in \Lambda U_s$.

Now $a - u$ is a cocycle in R_n. So a change of variables: $a' = a - u$ yields a new minimal Sullivan algebra $R_n' = (\Lambda U', \delta)$ where a minimal codegree even generator a' is a cocycle. □

Appendix A. Gorenstein rings and spaces

This appendix discusses h-Gorenstein spaces, emphasizing the duality this gives. The material comes from [17], [13] and [19], but the results have not been brought together explicitly before.

A.A. **Contents.** Recall that a commutative local Noetherian ring $(R, \mathfrak{m}, k)$ is *Cohen-Macaulay* if its depth is equal to its dimension, and that it is *Gorenstein* if R is of finite injective dimension as a module. Furthermore, if R is Gorenstein of dimension r

$$\mathrm{Ext}_R^*(k, R) = \mathrm{Ext}_R^r(k, R) = k.$$

Despite the definition, the important content of the Gorenstein condition is a duality property (this will be a special case of one in the CDGA case below).

Félix-Halperin-Thomas [17] have considered the analogue for spaces (which we call the h-Gorenstein condition) at length. We recall the definition below. It transpires that for spaces with finite dimensional cohomology (or finite category) X is h-Gorenstein if and only if $H^*(X)$ is Gorenstein. Our contribution is to make explicit the duality statements in the positive dimensional case following [13]. There is a structural duality statement at the level of derived categories even when $H^*(X)$ is not Gorenstein. Thus if X is h-Gorenstein, there are consequences for the cohomology ring [19]: the cohomology ring $H^*(X)$ is always generically Gorenstein, if it is Cohen-Macaulay, it is automatically Gorenstein (and hence its Hilbert series satisfies a functional equation), and if it has Cohen-Macaulay defect 1, its Hilbert series satisfies a suitable pair of functional equations.

A.B. **The definition.** We recall the definition from [17] in the language of [13].

Definition A.1. We say that a DGA A is *h-Gorenstein of shift a* if $\mathrm{Hom}_A(\mathbb{Q}, A) \simeq \Sigma^a \mathbb{Q}$. We say that a space X is *h-Gorenstein* if $C^*(X)$ is h-Gorenstein.

We begin with the remark that the definition is an invariant of quasi-isomorphism, so that any particular rational model of the space X may be used.

A.C. **Gorenstein duality.** The purpose of the Gorenstein condition is to capture a duality property. This takes some work to extract. Since the argument is in [13] we will be brief. Since we are now mixing two sorts of duality, it is essential to emphasize that the suspension Σ^a is *homological*: it increases degrees by a (i.e., it reduces codegrees by a).

Proposition A.2. [13] *If A is h-Gorenstein of shift a and $H^*(A)$ is 1-connected and Noetherian, then there is an equivalence*

$$\mathrm{Cell}_{\mathbb{Q}} A \simeq \Sigma^a A^\vee,$$

and hence a spectral sequence

$$H_I^*(H^*(A)) \Rightarrow \Sigma^a H^*(A)^\vee.$$

Proof: By definition, we have an equivalence $\mathrm{Hom}_A(\mathbb{Q}, A) \simeq \Sigma^a \mathbb{Q}$ of A-modules. To proceed we need to apply Morita theory, so we consider the endomorphism ring $\mathcal{E} = \mathrm{Hom}_A(\mathbb{Q}, \mathbb{Q})$. There is a natural right $\mathcal{E}$-module structure on $\mathrm{Hom}_A(\mathbb{Q}, M)$ for any M, so the Gorenstein condition gives an $\mathcal{E}$-action on $\mathbb{Q}$. However, since A is 1-connected, there is a unique $\mathcal{E}$-module structure on $\mathbb{Q}$. Thus the Gorenstein condition gives an equivalence

$$\mathrm{Hom}_A(\mathbb{Q}, A) \simeq \mathrm{Hom}_A(\mathbb{Q}, \Sigma^a A^\vee)$$

of $\mathcal{E}$-modules. Now apply $\otimes_{\mathcal{E}} \mathbb{Q}$. Since $H^*(A)$ is Noetherian, $\mathbb{Q}$ is proxy-small in the sense of [13], and we may use Morita theory to deduce

$$\mathrm{Cell}_{\mathbb{Q}}(A) \simeq \mathrm{Cell}_{\mathbb{Q}}(\Sigma^a A^\vee).$$

Since $A^\vee$ is $\mathbb{Q}$-cellular, the $\mathrm{Cell}_{\mathbb{Q}}$ on the right may be omitted.

This proves the first statement. For the second, by Corollary 4.5 the stable Koszul complex ΓA provides a model for $\mathrm{Cell}_{\mathbb{Q}}(A)$. Using the natural filtration, we obtain the spectral sequence. □

We remark that the spectral sequence collapses if $H^*(A)$ is Cohen-Macaulay to show $H_I^r(H^*(A)) \cong \Sigma^{a+r} H^*(A)^\vee$ (where r is the Krull dimension of $H^*(A)$). Thus $H^*(A)$ is also Gorenstein, and a is the classical a-invariant.

The spectral sequence also collapses if $H^*(A)$ is of Cohen-Macaulay defect 1, to give an exact sequence

$$0 \longrightarrow H_I^r(H^*(A)) \longrightarrow \Sigma^{a+r} H^*(A)^\vee \longrightarrow \Sigma H_I^{r-1}(H^*(A)) \longrightarrow 0.$$

This is discussed in more structural terms in [19, 5.4].

A.D. **Functional equations.** It may be helpful to record the functional equations satisfied by the Hilbert series $p_A(t)$ of an h-Gorenstein algebra A when $H^*(A)$ is a Noetherian ring of Cohen-Macaulay defect 0 or 1. The equations are deduced from the existence of a local cohomology theorem in [19, Section 6]. Since $H^*(A)$ is cograded, we take t to be of codegree 1 (i.e., of degree -1).

If $H^*(A)$ is Cohen-Macaulay we have

$$p_A(1/t) = (-t)^r t^a p_A(t).$$

If $H^*(A)$ is of Cohen-Macaulay defect 1, we have a pair of functional equations (introduced in the group theoretic context by Benson and Carlson [8])

$$p_A(1/t) - (-t)^r t^a p_A(t) = (-1)^{r-1}(1+t)\delta_A(t)$$

and

$$\delta_A(1/t) = (-t)^{r-1} t^a \delta_A(t),$$

and in fact $\delta_A(t)$ is the Hilbert series of $H_I^{r-1}(H^*(A))^\vee$.

A.E. **Examples.** First we show that there are many familiar examples of h-Gorenstein DGAs.

Corollary A.3. [17, 3.2(ii)] *If $H^*(A)$ is Gorenstein then A is h-Gorenstein.*

Proof: If $H^*(A)$ is Gorenstein then the E_2-term of the spectral sequence

$$\mathrm{Ext}^{*,*}_{H^*(A)}(\mathbb{Q}, H^*(A)) \Rightarrow H^*(\mathrm{Hom}_A(\mathbb{Q}, A))$$

degenerates to an isomorphism

$$\mathrm{Ext}^r_{H^*(A)}(\mathbb{Q}, H^*(A)) = \Sigma^{r+a}\mathbb{Q},$$

where a is the conventional a-invariant. The spectral sequence therefore collapses to show A is h-Gorenstein with shift a. □

Corollary A.4. [17, 3.6] *If $H^*(A)$ is finite dimensional then A is h-Gorenstein if and only if $H^*(A)$ is a Poincaré duality algebra.*

Proof: If $H^*(A)$ is a Poincaré duality algebra of formal dimension n then it is a zero dimensional Gorenstein ring with a-invariant $-n$, so A is h-Gorenstein with shift $-n$ by the previous corollary.

Conversely, if A is h-Gorenstein of shift a, we have a Gorenstein duality spectral sequence. Since $H^*(A)$ is finite dimensional, it is all torsion. Accordingly, $H_I^*(H^*(A)) = H^*(A)$, and the spectral sequence reads

$$H^*(A) = \Sigma^a H^*(A)^\vee$$

and $H^*(A)$ is a Poincaré duality algebra of formal dimension $-a$. □

One may use these to construct other examples which are h-Gorenstein but not Gorenstein.

Proposition A.5. [17, 4.3] *Suppose we have a fibration $F \longrightarrow E \longrightarrow B$ with F finite. If F and B are h-Gorenstein with shifts f and b then E is h-Gorenstein with shift $e = f + b$.* □

This allows us to construct innumerable examples. For example any finite Postnikov system is h-Gorenstein [17, 3.4], so that in particular any sci space is h-Gorenstein. A simple example will illustrate the duality.

Example A.6. We construct a rational space X in a fibration

$$S^3 \times S^3 \longrightarrow X \longrightarrow \mathbb{C}P^\infty \times \mathbb{C}P^\infty,$$

so that X is h-Gorenstein. We will calculate $H^*(X)$ and observe that it is not Gorenstein.

Let V be a graded vector space with two generators u, v in degree 2, and let W be a graded vector space with two generators in degree 4. The two 4-dimensional cohomology classes u^2, uv in $H^*(KV) = \mathbb{Q}[u, v]$ define a map $KV \longrightarrow KW$, and we let X be the fibre, so we have a fibration

$$S^3 \times S^3 \longrightarrow X \longrightarrow KV$$

as required. By [13], this is h-Gorenstein with shift -4 (being the sum of the shift (viz -6) of $S^3 \times S^3$ and the shift (viz 2) of KV).

It is amusing to calculate the cohomology ring of X. It is $\mathbb{Q}[u, v, p]/(u^2, uv, up, p^2)$ where u, v and p have degrees $2, 2$ and 5. The dimensions of its graded components are $1, 0, 2, 0, 1, 1, 1, 1, 1, \ldots$ (i.e., its Hilbert series is $p_X(t) = (1 + t^5)/(1 - t^2) + t^2$, where t is of codegree 1).

In calculating local cohomology it is useful to note that $\mathfrak{m} = \sqrt{(v)}$. The local cohomology is $H_I^0(H^*(X)) = \Sigma_2 \mathbb{Q}$ in degree 0 (so that $H^*(X)$ is not Cohen-Macaulay) and as a $\mathbb{Q}[v]$-module $H_I^1(H^*(X))$ is $\mathbb{Q}[v]^\vee \otimes (\Sigma^{-3}\mathbb{Q} \oplus \Sigma^2\mathbb{Q})$. Since there is no higher local cohomology the local cohomology spectral sequence necessarily collapses, and the resulting exact sequence

$$0 \longrightarrow H_I^1(H^*(X)) \longrightarrow \Sigma^{-4} H^*(A)^\vee \longrightarrow \Sigma^{-2}\mathbb{Q} \longrightarrow 0$$

is consistent.

Since the Cohen-Macaulay defect here is 1, we have a pair of functional equations

$$p_X(1/t) - (-t)t^{-4} p_X(t) = (1 + t)\delta(t)$$

and

$$\delta(1/t) = t^4 \delta(t).$$

Indeed, the first equation gives $\delta(t) = t^{-2}$, which is indeed the Hilbert series of $H_I^0(H^*(X))^\vee$, and it obviously satisfies the second equation.

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